Cartesian Equation of a Line in $\mathbb{R}^2$

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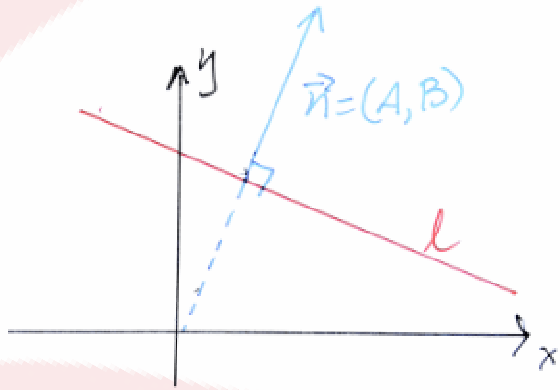


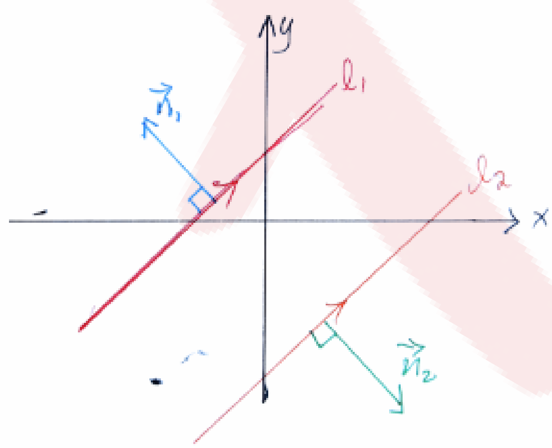
For a line l in$\mathbb{R}^2$ the *Cartesian equation of the line* is given by,

$$Ax + By + C = 0$$

where $\vec{n} = (A, B)$ is a *normal* to the line l. A *normal* is a vector perpendicular to the line l.

Let's consider parallel and perpendicular lines and how their normals may be related, if at all. Suppose l_1 and l_2 are two lines in $\mathbb{R}^2$ with normals $\vec{n_1}$ and $\vec{n_2}$, respectively.

Case 1: Parallel Lines Suppose l_1 and l_2 are parallel.



Since $l_1 \parallel l_2$ and $\overrightarrow{m_1} = k\overrightarrow{m_2}$ where $\overrightarrow{m_1}$ and $\overrightarrow{m_2}$ are direction vectors of l_1 and l_2, respectively. Therefore, $\overrightarrow{n_2} = k\overrightarrow{n_1}$. Why? We know the following,

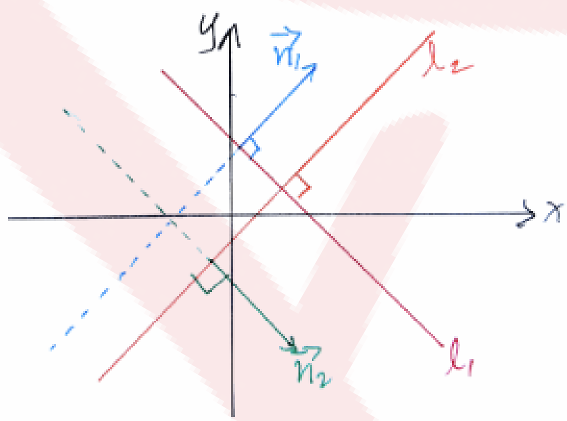
$$\overrightarrow{m} \cdot \overrightarrow{n} = 0 \quad \text{and} \quad \overrightarrow{m_2} \cdot \overrightarrow{n_2} = 0$$

From here we have,

$$\begin{aligned} 0 &= \overrightarrow{m_1} \cdot \overrightarrow{n_1} = (k\overrightarrow{m_2}) \cdot \overrightarrow{n_1} \\ &= k(\overrightarrow{m_2} \cdot \overrightarrow{n_1}) = \overrightarrow{m_2} \cdot (k\overrightarrow{n_1}) \\ \Rightarrow \overrightarrow{n_2} &= k\overrightarrow{n_1} \end{aligned}$$

So, the normals are multiples of each other.

Case 2: Perpendicular lines Suppose $l_1 \perp l_2$. Then $\overrightarrow{m_1} \cdot \overrightarrow{m_2} = 0$.



We know that $\overrightarrow{m_1} \cdot \overrightarrow{n_1} = 0 = \overrightarrow{m_2} \cdot \overrightarrow{n_2}$. This along with, $\overrightarrow{m_1} \cdot \overrightarrow{m_2} = 0$, we have,

$$\overrightarrow{m_1} \cdot \overrightarrow{n_1} - \overrightarrow{m_1} \cdot \overrightarrow{m_2} = 0, \quad \text{and} \tag{1}$$

$$\overrightarrow{m_2} \cdot \overrightarrow{n_2} - \overrightarrow{m_1} \cdot \overrightarrow{m_2} = 0 \tag{2}$$

From here we have,

$$\overrightarrow{m_1} \cdot (\overrightarrow{n_1} - \overrightarrow{m_2}) = 0, \text{ and} \tag{3}$$

$$\overrightarrow{m_2} \cdot (\overrightarrow{n_2} - \overrightarrow{m_1}) = 0, \quad \text{respectively.} \tag{4}$$

From (3) we can conclude that,

$$\begin{aligned} \overrightarrow{n_1} - \overrightarrow{m_2} = k\overrightarrow{n_1}, & \quad \text{for some } k \in \mathbb{R} \\ \Rightarrow (1-k)\overrightarrow{n_1} &= \overrightarrow{m_2} \end{aligned}$$

Similarly from equation (4) we have that $(1-k')\overrightarrow{n_2} = \overrightarrow{m_1}$. Therefore, since $\overrightarrow{m_1} \cdot \overrightarrow{m_2} = 0$ we have,

$$\begin{aligned} \overrightarrow{m_1} \cdot \overrightarrow{m_2} &= 0 \\ (1-k')\overrightarrow{n_2} \cdot (1-k)\overrightarrow{n_1} &= 0 \\ (1-k')(1-k)\overrightarrow{n_2} \cdot \overrightarrow{n_1} &= 0 \\ \Rightarrow \overrightarrow{n_2} \cdot \overrightarrow{n_1} &= 0 \end{aligned}$$

Therefore, the normals of l_1 and l_2 are perpendicular.

Example

Show that line $3x - 4y - 6 = 0$ and $6x - 8y + 12 = 0$ are parallel and distinct.

Solution: If we can show the normals of the lines are the same but the lines are different then we are done. The normal to line $l_1 : 3x - 4y - 6 = 0$ is $\overrightarrow{n_1} = (3, -4)$ and the normal to line $l_2 : 6x - 8y + 12 = 0$ is $\overrightarrow{n_2} = (6, -8)$. Since $\overrightarrow{n_2} = 2\overrightarrow{n_1}$, the two normals are the same and thus lines l_1 and l_2 are parallel. Let's consider the lines now. Notice that line l_2 can be simplified,

$$\begin{aligned} l_2 : 6x - 8y + 12 &= 0 \\ 3x - 4y + 6 &= 0, \quad \text{dividing both sides by 2.} \end{aligned}$$

Now we have,

$$\begin{aligned} l_2 : 3x - 4y + 6 &= 0, \quad \text{and} \\ l_1 : ex - 4y - 6 &= 0 \end{aligned}$$

Since lines l_1 and l_2 have a differenct C value, the lines l_1 and l_2 are distinct.

Example

For what value(s) of k are lines $l_1 : 3x-4y-6=0$ and $l_2 : kx+4y-5=0$ perpendicular?

Solution: The normal for l_1 is $\overrightarrow{n_1} = (3, -4)$ and for l_2 is $\overrightarrow{n_2} = (k, 4)$. If we can find a value(s) for k such that $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$ are prependicular then l_1 and l_2 are perpendicular. We can use the dot product to determine k since we know $\overrightarrow{n_1} \cdot \overrightarrow{n_2} = 0$

$$\begin{aligned} \overrightarrow{n_1} \cdot \overrightarrow{n_2} &= 0 \\ (3, -4) \cdot (k, 4) &= 0 \\ 3k - 16 &= 0 \\ 3k &= 16 \\ \therefore k &= \frac{16}{3} \end{aligned}$$

Example

What is the Cartesian equation of the line passing through (6,-2) and has normal $\overrightarrow{n} = (-1, 3)$.

Solution: The Carteisin form of the equation of a line is $Ax+By+C=0$ where (A, B) are the coordinates of the normal to the line. The normal in our case is $\overrightarrow{n} = (-1, 3)$ therefore, A=-1 and B=3. To find C, let's plug the point (6, -2) into what we have so far,

$$-x + 3y + C = 0.$$

$$\begin{aligned} -x+3y+C &= 0 \\ -6+3(-2)+C &= 0 \\ -6-6+C &= 0 \\ -12+C &= 0 \\ \therefore C &= 12 \end{aligned}$$

Therefore, the Cartesian equation of the line is given by,

$$-x+3y+12=0$$

Example

Given the vector equation of the line $\overrightarrow{r}=(3,-6)+s(-1,-4),\ s\in\mathbb{R}$, find

(a) The Cartesian equation

(b) parametric equation

(c) slope-intercept form of the line.

Solution:

(a) We ned to find a normal to the line. The direction vector is $\overrightarrow{m}=(-1,-4)$, the normal $\overrightarrow{n}=(A,B)$ is such that,

$$\begin{aligned} \overrightarrow{m}\cdot\overrightarrow{n} &= 0 \\ (-1,-4)\cdot(A,B) &= 0 \\ -A-4B &= 0 \\ A &= -4B \end{aligned}$$

Therefore, $\overrightarrow{n}=(-4B,B)=B(-4,1)$ where $B\in\mathbb{R}$ and we have

$$-4x+y+C=0. \tag{5}$$

We now need to find C. We know that (3, -6) is a point on the line so let's use this to find C by plugging it into equation (5).

$$\begin{aligned}
-4x + y + C &= 0 \\
-4(3) + (-6) + C &= 0 \\
-12 - 6 + C &= 0 \\
-18 + C &= 0 \\
C &= 18
\end{aligned}$$

Therefore, our Cartesian equation of the line is $-4x + y + 18 = 0$.

(b) The parametric equation is given by,

$$\begin{aligned}
x &= 3 - x \\
y &= -6 - 4s, \ s \in \mathbb{R}
\end{aligned}$$

(c) The slope can be determined from the direction vector $\overrightarrow{m} = (-1, -4)$. Slope is calculated as follows,

$$\begin{aligned}
m &= \frac{rise}{run} \\
&= \frac{-4}{-1} = 4
\end{aligned}$$

To find the y-intercept let's plug in a point we know lies on the line into our slope-intercept form and solve for b. We know (3,-6) is a point on the line so let's use this point.

$$\begin{aligned}
y &= mx + b \\
y &= 4x + b, \text{ since the slope is 4} \\
-6 &= 4(3) + b, \text{ from plugging in (3, -6) into the above} \\
-6 &= 12 + b \\
-18 &= b
\end{aligned}$$

Therefore, our slope-intercept form of the line is

$$y = 4x - 18.$$

Exercises

1. Given the line $y = -\frac{5}{6}x + 9$ find the following,
   (a) the direction vector a line parallel to the given line
   (b) the direction vector for a line that is perpendicular to the given line
   (c) the coordinates of a point on the given line
   (d) the vector and parametric form of the equation of a line *parallel* to the given line passing through A(7,9)
   (e) the vector and parametric form of the equation of a line *perpendicular* to the given line pasising through B(-2, 1).
2. For each of the given lines, determine the vector and parametric equations.
   (a) $y = \frac{7}{8}x - 6$
   (b) $x = 4$
3. Show that the following lines are coincident, $x - 3y + 4 = 0$ and $6x - 18y + 24 = 0$.
4. Two lines have equations $2x - 3y + 6 = 0$ and $4x = 6y = k = 0$.
   (a) Explain using normal vectors why these lines are parallel.
   (b) For what values of k will these lines be coincident.
5. Determin the Cartesian equation for the line with a normal vector of (4,5) passing through the point A(-1,5).
6. A line passes through the points A(-3,5) and B(-2,4). Determine the Cartesian equation of this line.
7. A line is perpendicular to the line $2x - 4y + 7 = 0$ and that passes through the point P(7, 2). Determine the equation of this line in Cartesian form.

8. The angle between any pair of lines in Cartesian form is also the angle between their nromal vectors. For the lines $x - 3y + 6 = 0$ and $x + 2y - 7 = 0$ determine the acute and obtuse angles between these two lines.

9. For each pair of lines, determine the size of the acute angle to the nearest degree that is created by the intersection of the line.
   (a) $(x, y) = (3, 6) + t(2, -5)$ and $(x, y) = (-3, 4) - t(-4, -1)$
   (b) $y = 0.5x + 6$ and $y = -0.75x - 1$
   (c) $x - 2t, y = 1 - 5t$ and $(x, y) = (4.0) + t(-4, 1)$.