Scalar and Vector Projections

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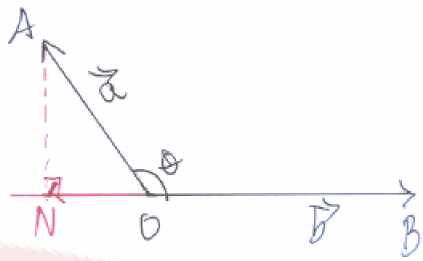
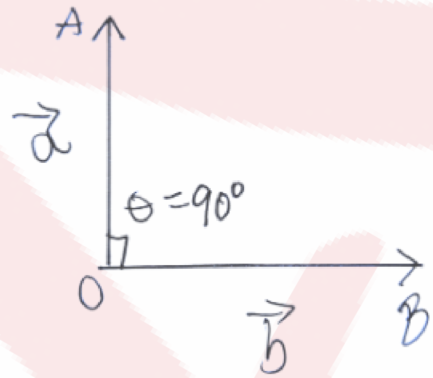
Scalar and Vector Projections

Given vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ the *scalar projection* of $\overrightarrow{a}$ onto $\overrightarrow{b}$ is $\overrightarrow{ON} = |\overrightarrow{a}|\cos\theta$, given the diagram below,





If we use the geometric defintion of the dot project we have the following,

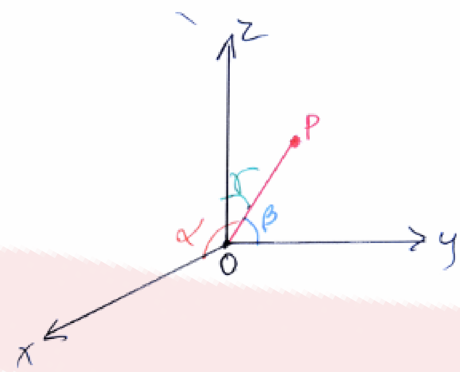
$$\begin{aligned} \overrightarrow{a} \cdot \overrightarrow{b} &= |\overrightarrow{a}||\overrightarrow{b}|\cos\theta \\ \overrightarrow{a} \cdot \overrightarrow{b} &= (|\overrightarrow{a}|\cos\theta)|\overrightarrow{b}| \\ \frac{\overrightarrow{a} \cdot \overrightarrow{b}}{|\overrightarrow{b}|} &= |\overrightarrow{a}|\cos\theta \end{aligned}$$

Therefore,

$$\frac{\overrightarrow{a} \cdot \overrightarrow{b}}{|\overrightarrow{b}|} = |\overrightarrow{ON}|$$

is another way to calculate the scalar projection of $\overrightarrow{a}$ onto $\overrightarrow{b}$.

Direction Cosines for $\overrightarrow{a} = (a, b, c)$



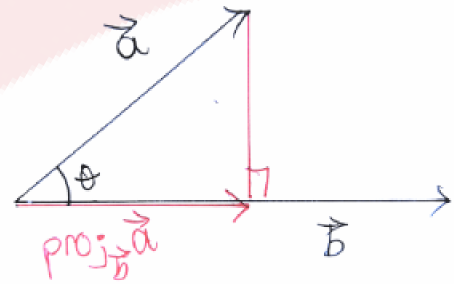
Give a vector $\overrightarrow{a} - \overrightarrow{OP}$ the *direction cosines* are the angles $\overrightarrow{OP}$ makes with the x, y and z-axes. let α, β, γ be the angles $\overrightarrow{a} = \overrightarrow{OP}$ makes with the axes x, y and z, repsectively then,

$$\begin{aligned}
\cos\alpha &= \frac{a}{|\overrightarrow{OP}|} = \frac{a}{\sqrt{a^2+b^2+c^2}} \\
\cos\beta &= \frac{b}{\sqrt{a^2+b^2+c^2}} \\
\cos\gamma &= \frac{c}{\sqrt{a^2+b^2+c^2}}
\end{aligned}$$

Vector Projections

The vector projection of vector $\overrightarrow{a}$ onto $\overrightarrow{b}$ is given by,

$$\begin{aligned}
proj_{\overrightarrow{b}}\overrightarrow{a} &= \left(\frac{\overrightarrow{a}\cdot\overrightarrow{b}}{|\overrightarrow{b}|^2}\right)\overrightarrow{b}, \quad \overrightarrow{b} \neq \overrightarrow{0} \\
&= \left(\frac{\overrightarrow{a}\cdot\overrightarrow{b}}{\overrightarrow{b}\cdot\overrightarrow{b}}\right)\overrightarrow{b}
\end{aligned}$$



Example

For vectors $\overrightarrow{a} = (-3, 4, 5\sqrt{3})$, $\overrightarrow{b} = (-2, 2, -1)$.

(a) Find the scalar projection of $\overrightarrow{a}$ onto $\overrightarrow{b}$.

(b) Find the vector projection of $\overrightarrow{b}$ onto $\overrightarrow{a}$.

(c) Find the direction cosines for $\overrightarrow{b}$.

Solution:

(a)

$$\begin{aligned} |\overrightarrow{a}|\cos\theta &= \frac{\overrightarrow{a}\cdot\overrightarrow{b}}{|\overrightarrow{b}|} \\ &= \frac{(-3, 4, 5\sqrt{3})\cdot(-2, 2, -1)}{\sqrt{4+4+1}} \\ &= \frac{6+8-5\sqrt{3}}{\sqrt{9}} \\ &= \frac{14-5\sqrt{3}}{3} \end{aligned}$$

(b)

$$
\begin{aligned}
proj_{\overrightarrow{b}}\overrightarrow{a} &= \left(\frac{\overrightarrow{b}\cdot\overrightarrow{a}}{\overrightarrow{a}\cdot\overrightarrow{a}}\right)\overrightarrow{a} \\
&= \frac{(-2,2,-1)\cdot(-3,4,5\sqrt{3})}{\sqrt{9+16+25(3)}}(-3,4,5\sqrt{3}) \\
&= \frac{(6+8-5\sqrt{3}}{\sqrt{25+75}}(-3,4,5\sqrt{3}) \\
&= \frac{14-5\sqrt{3}}{10}(-3,4,5\sqrt{3})
\end{aligned}
$$

(c)

$$
\begin{aligned}
\cos\alpha &= \frac{-2}{\sqrt{4+4+1}} = \frac{2}{3} \\
\cos\beta &= \frac{2}{3} \\
\cos\gamma &= -\frac{1}{3}
\end{aligned}
$$

Exercises

1. (a) The vector $\overrightarrow{a} = (2, 3)$ is projected onto the x-axis. What is the scalar projection? What is the vector projection?

   (b) What are the scalar and vector projections when $\overrightarrow{a}$ is projected onto the y-axis?

2. Determine the scalr and vector projections of $\overrightarrow{OP} = (-1, 2, -5)$ on $\hat{i}, \hat{j}$ and $\hat{k}$.

3. For each of the following derermine the scalar and vector projections of $\overrightarrow{x}$ on $\overrightarrow{y}$.

   (a) $\overrightarrow{x} = (1, 1)$, $\overrightarrow{y} = (1, -1)$

   (b) $\overrightarrow{x} = (2, 2\sqrt{3})$, $\overrightarrow{y} = (1, 0)$

   (c) $\overrightarrow{x} = (2, 5)$, $\overrightarrow{y} = (-5, 12)$

4. (a) Using a diagram show that the vector projection of $-\overrightarrow{a}$ on $\overrightarrow{a}$ is $-\overrightarrow{a}$.

   (b) Using the formula for scalar projections, show that the result in (a) is true.

5. (a) Find the scalar and vector projections of $\overrightarrow{AB}$ along each of the axes if A has coordinates $(1, 2, 2)$ and B = (-1, 3, 4).

   (b) What angle does $\overrightarrow{AB}$ make with the y-axis?

6. Vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ are such that $|\overrightarrow{a}| = 10$ and $|\overrightarrow{b}| = 12$ and the angle between them is $135°$.

   (a) Show that the scalr projection of $\overrightarrow{a}$ on $\overrightarrow{b}$ does not equal the scalar projection of $\overrightarrow{b}$ on $\overrightarrow{a}$.

   (b) Draw diagrams to illustrate the corresponding vector projections associated with (a).

7. Given $\overrightarrow{OD} = (-1, 2, 2)$ and the points A(-2, 1, 4), B(1, 3, 3) and C(-6, 7, 5), calculate the scalar projection of $\overrightarrow{AB}$ on $\overrightarrow{OD}$.

8. If α, β and γ represent the direction angles for $\overrightarrow{OP}$, prove the following,

   (a)
   $$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

   (b)
   $$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$$