Cross Product of Two Vectors

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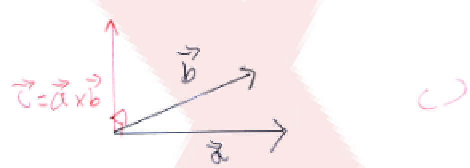
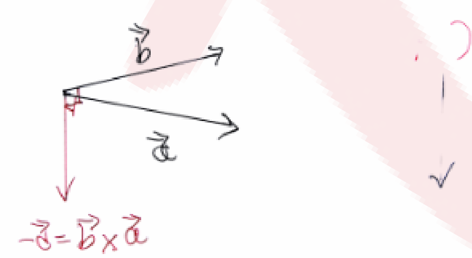
Cross Product of Two Vectors

The *cross product* or *vector product* of two vectors of $\mathbb{R}^3$ results in a vector that is perpendicular to the two given vectors, $\overrightarrow{a}$ and $\overrightarrow{b}$, say. For example, given the two vectors, $\overrightarrow{a} = (1,1,1)$ and $\overrightarrow{b} = (1,2,-1)$ we want to determine the cross product, $\overrightarrow{c} = \overrightarrow{a} \times \overrightarrow{b}$.

$$\begin{aligned}
\overrightarrow{a} \times \overrightarrow{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 2 & -1 \end{vmatrix} \\
&= ((1)(-1) - (1)(2))\hat{i} - ((1)(-1) - (1)(1))\hat{j} + ((1)(2) - (1)(1))\hat{k} \\
&= (-1-2)\hat{i} - (-1-1)\hat{j} + (2-1)\hat{k} \\
&= -3\hat{i} + 2\hat{j} + \hat{k} \\
&= -3(1,0,0) + 2(0,1,0) + (0,0,1) \\
&= (-3,2,1) = \overrightarrow{c}
\end{aligned}$$

Determining the direction of the cross product





Given the cross product $\overrightarrow{c} = \overrightarrow{a} \times \overrightarrow{b}$, and the first diagram above, curl you fingers in the direction from $\overrightarrow{a}$ to $\overrightarrow{b}$ and the direction your thumb points, is the direction of the cross product $\overrightarrow{c} = \overrightarrow{a} \times \overrightarrow{b}$.

Formula for $\overrightarrow{a} \times \overrightarrow{b}$

Given $\overrightarrow{a} =)a_1, a_2, a_3)$ and $\overrightarrow{b} = (b_1, b_2, b_3)$ then $\overrightarrow{a} \times \overrightarrow{b}$ is given by,

$$(a_2b_2 - a_3b_3 - (a_1b_3 - a_3b_1), a_1b_2 - a_2b_1)$$

Notice in the first example that this formula was used.

An easy way to remember ths formula for the cross product of two vectors is below:

Write $\overrightarrow{a}$ and $\overrightarrow{b}$ in a "table" as follows,

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

where $\hat{i} = (1, 0, 0), \hat{j} = (0, 1, 0)$ and $\hat{k} = (0, 0, 1)$ are the coordinate vectors. Then,

$$\overrightarrow{a} \times \overrightarrow{b} = \hat{i}(a_2b_2 - a_3b_3) - \hat{j}(a_1b_3 - a_3b_1) + \hat{k}(a_1b_2 - a_2b_1)$$

Properties of the Cross Product

Let $\overrightarrow{a}, \overrightarrow{b}$ and $\overrightarrow{c}$ be three vectors in $\mathbb{R}^3$ and $k \in \mathbb{R}$. Then,

1. $\overrightarrow{a} \times \overrightarrow{b} = -\overrightarrow{b} \times \overrightarrow{a}$
2. $\overrightarrow{a} \times (\overrightarrow{b} + \overrightarrow{c}) = \overrightarrow{a} \times \overrightarrow{b} + \overrightarrow{a} \times \overrightarrow{c}$
3. $k(\overrightarrow{a} \times \overrightarrow{b}) = (k\overrightarrow{a}) \times \overrightarrow{b} = \overrightarrow{a} \times (k\overrightarrow{b})$

Example

For $\overrightarrow{p} = (-1, 3, 2)$ and $\overrightarrow{q} = (2, -5, 6)$ find $\overrightarrow{p} \times \overrightarrow{q}$.

Solution:

$$
\begin{aligned}
\overrightarrow{p} \times \overrightarrow{q} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 3 & 2 \\ 2 & -5 & 6 \end{vmatrix} \\
&= \hat{i}((3)(6) - (-5)(2)) - \hat{j}((-1)(6) - (2)(2)) + \hat{k}((-1)(-5) - (2)(3)) \\
&= \hat{i}(18 + 10) - \hat{j}(-6 - 4) + \hat{k}(5 - 6) \\
&= 28\hat{i} + 10\hat{j} - \hat{k} \\
&= (28, 10, -1)
\end{aligned}
$$

Exercises

1. For each of the following calculations, state which are possible for vectors in $\mathbb{R}^3$ and which are not. Explain.
   (a) $\overrightarrow{a} \cdot (\overrightarrow{b} \times \overrightarrow{c})$
   (b) $(\overrightarrow{a} \cdot \overrightarrow{b}) \times \overrightarrow{c}$
   (c) $(\overrightarrow{a} \times \overrightarrow{b}) \cdot (\overrightarrow{c} + \overrightarrow{d})$
   (d) $(\overrightarrow{a} \cdot \overrightarrow{b})(\overrightarrow{c} \times \overrightarrow{d})$
   (e) $(\overrightarrow{a} \times \overrightarrow{b}) \times (\overrightarrow{c} \times \overrightarrow{d})$
   (f) $\overrightarrow{a} \times \overrightarrow{b} + \overrightarrow{c}$

2. Calculate the cross product for each of the following pairs of vectors and verify the answer using the dot product.
   (a) (2, -3, 5) and (0, -1, 4)
   (b) (5, -1, 1) and (2, 4, 7)
   (c) (-2, 3, 3) and (1, -1, 0)

3. If $(-1, 3, 5) \times (0, a, 1) = (-2, 1, -1)$, determine a.

4. (a) For the vectors (1, 2, 1) and (2, 4, 2) show that their vector product is $\overrightarrow{0}$.
   (b) In general show that the vector product of two collinear vecors (a, b, c) and (ka, kb, kc) is always $\overrightarrow{0}$.

5. Verify each of the following,
   (a) $\hat{i} \times \hat{j} = \hat{k} = -\hat{j} \times \hat{i}$
   (b) $\hat{j} \times \hat{k} = \hat{i} = -\hat{k} \times \hat{j}$
   (c) $\hat{k} \times \hat{i} = \hat{j} = -\hat{i} \times \hat{k}$

6. Show algebraically that

$$k(a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1) \cdot \overrightarrow{a} = 0.$$

What is the meaning of this result?

7. Given the vectors $\overrightarrow{a} = (2,0,0)$, $\overrightarrow{b} = (0,3,0)$, $\overrightarrow{c} = (2,3,0)$ and $\overrightarrow{d} = (4,3,0)$.

   (a) Calculate $\overrightarrow{a} \times \overrightarrow{b}$ and $\overrightarrow{c} \times \overrightarrow{d}$.

   (b) Calculate $(\overrightarrow{a} \times \overrightarrow{b}) \times (\overrightarrow{c} \times \overrightarrow{d})$.

   (c) Without doing any calculations, say why $(\overrightarrow{a} \times \overrightarrow{c}) \times (\overrightarrow{b} \times \overrightarrow{d}) = 0$

8. Prove that $(\overrightarrow{a} - \overrightarrow{b}) \times (\overrightarrow{a} + \overrightarrow{b}) = 2\overrightarrow{a} \times \overrightarrow{b}$ is true.