Dot Product of two Vectors

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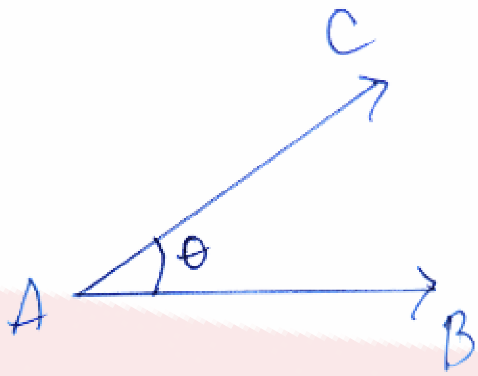
Let's define the *dot product* and then look at its properties.



The *dot product* of vectors $\overrightarrow{AC}$ and $\overrightarrow{AB}$ is given by,

$$\overrightarrow{AC} \cdot \overrightarrow{AB} = |\overrightarrow{AC}||\overrightarrow{AB}| \cos\theta, \ 0 \leq \theta \leq 180$$

The dot product is always a scalar and so may be referred to as the *scalar product*. The dot product can take on any real value, positive, negative or zero, since $\cos\theta$ can be any real value.

For vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ the dot product has the following properties,

$$\overrightarrow{a} \cdot \overrightarrow{b} = |\overrightarrow{a}||\overrightarrow{b}| \cos\theta$$

we have,

$$\begin{aligned} 0 \leq \theta < 90 \ &\Rightarrow \ \overrightarrow{a} \cdot \overrightarrow{b} > 0 \\ \theta = 90 \ &\Rightarrow \ \overrightarrow{a} \cdot \overrightarrow{b} = 0 \\ 90 < \theta \leq 180 \ &\Rightarrow \ \overrightarrow{a} \cdot \overrightarrow{b} < 0 \end{aligned}$$

Properites of the Dot Product

For vectors $\overrightarrow{p}$, $\overrightarrow{q}$ and $\overrightarrow{r}$ we have,

$$\begin{aligned}\overrightarrow{p}\cdot\overrightarrow{q} &= \overrightarrow{q}\cdot\overrightarrow{p} &(1)\\ \overrightarrow{p}\cdot(\overrightarrow{q}+\overrightarrow{r}) &= \overrightarrow{p}\cdot\overrightarrow{q}+\overrightarrow{p}\cdot\overrightarrow{r} &(2)\\ \overrightarrow{p}\cdot\overrightarrow{p} &= |\overrightarrow{p}|^2 &(3)\\ \text{For scalar } k,\ (k\overrightarrow{p})\cdot\overrightarrow{q} &= \overrightarrow{p}\cdot(k\overrightarrow{q}) = k(\overrightarrow{p}\cdot\overrightarrow{q}) &(4)\end{aligned}$$

Example

Vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ are placed tail to tail, have magnitude 4 and 7, respectively. The angle between $\overrightarrow{a}$ and $\overrightarrow{b}$ is 6-°. What is $\overrightarrow{a}\cdot\overrightarrow{b}$?

Solution:

$$\begin{aligned}\overrightarrow{a}\cdot\overrightarrow{b} &= |\overrightarrow{a}||\overrightarrow{b}|\cos\theta\\ &= (4)(7)\cos 60^\circ\\ &= 28(1/2)\\ &= 14\end{aligned}$$

Example

Prove the following: If $|\overrightarrow{x}+\overrightarrow{y}| = |\overrightarrow{x}-\overrightarrow{y}|$ then $\overrightarrow{x}\cdot\overrightarrow{y} = 0$.

Solution:

$$
\begin{aligned}
|\overrightarrow{x}+\overrightarrow{y}| &= |\overrightarrow{x}-\overrightarrow{y}| \\
|\overrightarrow{x}+\overrightarrow{y}|^2 &= |\overrightarrow{x}-\overrightarrow{y}|^2 \\
(\overrightarrow{x}+\overrightarrow{y})\cdot(\overrightarrow{x}+\overrightarrow{y}) &= (\overrightarrow{x}-\overrightarrow{y})\cdot(\overrightarrow{x}-\overrightarrow{y}) \\
\overrightarrow{x}\cdot\overrightarrow{x}+\overrightarrow{x}\cdot\overrightarrow{y}+\overrightarrow{y}\cdot\overrightarrow{x}+\overrightarrow{y}\cdot\overrightarrow{y} &= \overrightarrow{x}\cdot\overrightarrow{x}-\overrightarrow{x}\cdot\overrightarrow{y}-\overrightarrow{y}\cdot\overrightarrow{x}+\overrightarrow{y}\cdot\overrightarrow{y} \\
|\overrightarrow{x}|^2+\overrightarrow{x}\cdot\overrightarrow{y}+\overrightarrow{y}\cdot\overrightarrow{x}+|\overrightarrow{y}|^2 &= |\overrightarrow{x}|^2-\overrightarrow{x}\cdot\overrightarrow{y}-\overrightarrow{y}\cdot\overrightarrow{x}+|\overrightarrow{y}|^2 \\
2\overrightarrow{x}\cdot\overrightarrow{y} &= -2\overrightarrow{x}\cdot\overrightarrow{y} \\
\overrightarrow{x}\cdot\overrightarrow{y} &= -\overrightarrow{x}\cdot\overrightarrow{y} \\
\overrightarrow{x}\cdot\overrightarrow{y}+\overrightarrow{x}\cdot\overrightarrow{y} &= 0 \\
2\overrightarrow{x}\cdot\overrightarrow{y} &= 0 \\
\overrightarrow{x}\cdot\overrightarrow{y} &= 0 \\
\therefore \cos\theta = 0 \quad &\text{or} \quad \theta = 90^\circ.
\end{aligned}
$$

Unit Vectors

A *unit vector* is a vector $\overrightarrow{u}$ with magnitude equal 1, $|\overrightarrow{u} = 1$. For example, all the coorindate vectors $\hat{i} = (1,0,0), \hat{j} = (0,1,0), \hat{k} = (0,0,1)$ are all unit vectors.

Example

Show that $\hat{i}, \hat{j}$ and $\hat{k}$ are unit vectors using the definition of the dot product.

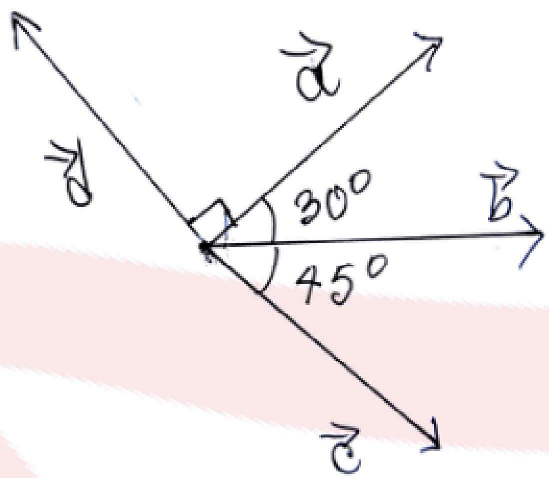
Solution:

$$
\begin{aligned}
\hat{i}\cdot\hat{i} &= |\hat{i}||\hat{i}|\cos\theta \\
&= (1)(1)\cos 0^\circ \\
&= 1
\end{aligned}
$$

Similary for $\hat{j}$ and $\hat{k}$.

Exercises

1. Given vectors $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$ and $\overrightarrow{d}$ where $|\overrightarrow{a}| = 5, |\overrightarrow{b}| = 3, |\overrightarrow{c}| = 4$ and $|\overrightarrow{d}| = 2$, angles between $\overrightarrow{a}$, $\overrightarrow{b}$ and $\overrightarrow{c}$ are given below,



Find the following values,

(a) $\overrightarrow{a} \cdot \overrightarrow{b}$

(b) $\overrightarrow{b} \cdot \overrightarrow{a}$

(c) $\overrightarrow{c} \cdot (\overrightarrow{a} + \overrightarrow{b})$

(d) $\overrightarrow{d} \cdot \overrightarrow{d}$

(e) $(3\overrightarrow{a}) \cdot \overrightarrow{c}$

(f) $\overrightarrow{d} \cdot \overrightarrow{d}$

(g) $\overrightarrow{b} \cdot \overrightarrow{0}$

2. If two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ are unit vectors pointing in opposite directions, what is the value of $\overrightarrow{a} \cdot \overrightarrow{b}$?

3. If θ is the angle in degrees between the two given vectors calculate the dot product of the vectors.

(a) $|\overrightarrow{x}| = 2, |\overrightarrow{y} = 4, \theta = 150^\circ$

(b) $|\overrightarrow{p}| = 1, |\overrightarrow{q} = q, \theta = 180^\circ$

(c) $|\overrightarrow{u}| = 4, |\overrightarrow{v} = 8, \theta = 145^\circ$

4. Calculate to the nearest degree the angle between the given vectors.

   (a) $|\overrightarrow{x}| = 2, |\overrightarrow{y} = 4, \overrightarrow{x} \cdot \overrightarrow{y} = 12\sqrt{3}$

   (b) $|\overrightarrow{p}| = 1, |\overrightarrow{q} = 5, \overrightarrow{p} \cdot \overrightarrow{q} = 3$

   (c) $|\overrightarrow{a}| = 7, |\overrightarrow{b} = 3, \overrightarrow{a} \cdot \overrightarrow{b} = 10.5$

5. Use the properites of the dot product to simplify each of the following expression,

   (a) $(\overrightarrow{a} + 5\overrightarrow{b}) \cdot (2\overrightarrow{a} - 3\overrightarrow{b})$

   (b) $3\overrightarrow{x} \cdot (\overrightarrow{x} - 3\overrightarrow{y}) - (\overrightarrow{x} - 3\overrightarrow{y}) \cdot (-3\overrightarrow{x} + \overrightarrow{y})$

6. The vectors $\overrightarrow{a} - 5\overrightarrow{b}$ and $\overrightarrow{a} - \overrightarrow{b}$ are perpendicular. If $\overrightarrow{a}$ and $\overrightarrow{b}$ are unit vectors, then determine $\overrightarrow{a} \cdot \overrightarrow{b}$.

7. If $\overrightarrow{a}$ and $\overrightarrow{b}$ are any two nonzero vectors prove the following,

$$(\overrightarrow{a} + \overrightarrow{b}) \cdot (\overrightarrow{a} + \overrightarrow{b}) = |\overrightarrow{a}|^2 + 2\overrightarrow{a} \cdot \overrightarrow{b} + |\overrightarrow{b}|^2$$

8. Let $\overrightarrow{u}$, $\overrightarrow{v}$ and $\overrightarrow{w}$ be three mutually perpendicular vectors of lengths 1, 2 ad 3, respectively. Calculate the value of $(\overrightarrow{u} + \overrightarrow{v} + \overrightarrow{w}) \cdot (\overrightarrow{u} + \overrightarrow{v} + \overrightarrow{w})$.

9. The vector $\overrightarrow{a}$ is a unit vector and the vector $\overrightarrow{b}$ is any other nonzero vector. If $\overrightarrow{c} = (\overrightarrow{b} \cdot \overrightarrow{a})\overrightarrow{a}$ and $\overrightarrow{d} = \overrightarrow{b} - \overrightarrow{c}$ prove that $\overrightarrow{d} \cdot \overrightarrow{a} = 0$.

