Dot Product of two Vectors (Algebraically)

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For $\overrightarrow{a}, \overrightarrow{b} \in \mathbb{R}^3$ where $\overrightarrow{a} = (a_1, a_2, a_3)$ and $\overrightarrow{b} = (b_1, b_2, b_3)$ then,

$$\overrightarrow{a} \cdot \overrightarrow{b} = \overrightarrow{b} \cdot \overrightarrow{a} = a_1b_1 + a_2b_2 + a_3b_3 = b_1a_1 + b_2a_2 + b_3a_3$$

Similarly for $\overrightarrow{a}, \overrightarrow{b} \in \mathbb{R}^2$ where $\overrightarrow{a} = (a_1, a_2)$ and $\overrightarrow{b} = (b_1, b_2)$ then,

$$\overrightarrow{a} \cdot \overrightarrow{b} = \overrightarrow{b} \cdot \overrightarrow{a} = a_1b_1 + a_2b_2 = b_1a_1 + b_2a_2$$

Using the above algebraic formulas for the dot product we have the following formulas for the product,

(a) In $\mathbb{R}^2$,

$$\overrightarrow{x} \cdot \overrightarrow{y} = |\overrightarrow{x}||\overrightarrow{y}| \cos\theta = x_1y_1 + x_2y_2 \qquad (1)$$

where $\overrightarrow{x} = (x_1, x_2)$ and $\overrightarrow{y} = (y_1, y_2)$ and θ is the angle between $\overrightarrow{x}$ and $\overrightarrow{y}$.

(b) In $\mathbb{R}^3$,

$$\overrightarrow{x} \cdot \overrightarrow{y} = |\overrightarrow{x}||\overrightarrow{y}| \cos\theta = x_1y_1 + x_2y_2 + x_3y_3 \qquad (2)$$

where $\overrightarrow{x} = (x_1, x_2, x_3)$ and $\overrightarrow{y} = (y_1, y_2, y_3)$ and θ is the angle between $\overrightarrow{x}$ and $\overrightarrow{y}$.

Using the relationships in equation (1) and (2) we can solve for the angle between $\overrightarrow{x}$ and $\overrightarrow{y}$.

Example

(a) What is the angle between vector $\overrightarrow{a} = (-1, 3, 4)$ and $\overrightarrow{b} = (2, 1, -3)$?

(b) For what values of m are the vectors $\overrightarrow{x} = (m, m-3)$ and $\overrightarrow{y} = (m, -3, 6)$ perpendicular?

(c) Determine the angle between $\overrightarrow{a} = (5, -1)$ and $\overrightarrow{b} = (2, 3)$?

(d) Find a vector perpendicular to $\overrightarrow{x} = (1, 4, -1)$ and $\overrightarrow{y} = (3, -1, 2)$.

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Solution:

(a)

$$\begin{aligned}\overrightarrow{a}\cdot\overrightarrow{b} &= |\overrightarrow{a}||\overrightarrow{b}|\cos\theta = a_1b_1+a_2b_2+a_3b_3\\(\sqrt{1+9+16})(\sqrt{4+1+9})\cos\theta &= (-1,3,4)\cdot(2,1,-3)\\\cos\theta &= \frac{-2+3-12}{\sqrt{26}\sqrt{14}}\\\cos\theta &= \frac{-11}{\sqrt{26}\sqrt{14}}\end{aligned}$$

(b)

$$\begin{aligned}\overrightarrow{x}\cdot\overrightarrow{y} &= 0\\(m,m,-3)\cdot(m,-3,6) &= 0\\m^2-3m-18 &= 0\\(m-6)(m+3) &= 0\end{aligned}$$

Therefore, $m=6$ or $m=-3$.

(c)

$$\begin{aligned}\overrightarrow{a}\cdot\overrightarrow{b} &= |\overrightarrow{a}||\overrightarrow{b}|\cos\theta\\(5,-1)\cdot(2,3) &= \sqrt{25+1}\sqrt{4+9}\cos\theta\\10-3 &= \sqrt{26}\sqrt{13}\cos\theta\\\frac{7}{\sqrt{26}\sqrt{13}} &= \cos\theta\end{aligned}$$

(d) Let $\overrightarrow{a}=(a,b,c)$ be perpendicular to $\overrightarrow{x}=(1,4,-1)$ and $\overrightarrow{y}=(3,-1,2)$.

$$\begin{aligned}\overrightarrow{a}\cdot\overrightarrow{x} &= 0 \text{ and}\\\overrightarrow{a}\cdot\overrightarrow{y} &= 0\end{aligned}$$

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giving,

$$a + 4b - c = 0 \quad (3)$$
$$3a - b + 2c = 0 \quad (4)$$

Multiplying equation (3) by 3 and subtracting equation (4) from this we get,

$$3 \times (3) \Rightarrow 3a + 12b - 3c = 0 \quad (5)$$
$$3a - b + 2c = 0 \quad (6)$$
$$(7)$$

(5) - (6) gives,

$$13b - 5c = 0$$

Therefore, $c = \frac{13b}{5}$. Plugging this expression for c into equation (3) we get,

$$\begin{aligned} a + 4b - c &= 0 \\ a + 4b - \frac{13b}{5} &= 0 \\ 5a + 10b - 13b &= 0 \\ 5a + 7b &= 0 \\ \therefore a &= \frac{7b}{-5} \end{aligned}$$

Therefore, we have,

$$\overrightarrow{a} = \left(\frac{7b}{-5}, b, \frac{13b}{5}\right) = b\left(-\frac{7}{5}, 1, \frac{13}{5}\right)$$

If we take $b = 5$ then we have, $\overrightarrow{a} = (-7, 5, 13)$ which is perpendicular to $\overrightarrow{x}$ and $\overrightarrow{y}$.

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Exercises

1. How many vectors are perpendicular to $\overrightarrow{a} = (-1, 1)$? State the components of three such vectors.

2. For each of the following pairs of vectors, calculate the dot product and, state whether the angle between the vectors is acute, obtuse or a right angle.
   (a) $\overrightarrow{a} = (-2, 1)$, $\overrightarrow{b} = (1, 2)$
   (b) $\overrightarrow{a} = (2, 3, -1)$, $\overrightarrow{b} = (4, 3, -17)$
   (c) $\overrightarrow{a} = (1, -2, 5)$, $\overrightarrow{b} = (3, -2, -2)$

3. Determine the angle to the nearest degree between each of the following pairs of vectors,
   (a) $\overrightarrow{a} = (5, 3)$, $\overrightarrow{b} = (-1, -2)$
   (b) $\overrightarrow{a} = (2, 2, 1)$, $\overrightarrow{b} = (2, 1, -2)$

4. Determine the value k given $\overrightarrow{a} = (-1, 2, -3)$, $\overrightarrow{b} = (-6k, -1, k)$, $\theta = 45°$.

5. Determine the angle, to the nearest degree for the vectors $\overrightarrow{a} = (\sqrt{2} - 1, \sqrt{2} + 1, \sqrt{2})$ and $\overrightarrow{b} = (1, 1, 1)$.

6. (a) For the vectors $\overrightarrow{a} = (2, p, 8)$ and $\overrightarrow{b} = (q, 4, 12)$ determine values of p and q so that the vectors are,
      (a) collinear
      (b) perpendicular
   (c) Are the values of p and q unique? Explain why or why not.

7. ΔABC has vertices A(2,5), B(4, 11) and C(-1,6). Determine the angles in this triangle.

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8. The vectors $\overrightarrow{x} = (-4, p, -2)$ and $\overrightarrow{y} = (-2, 3, 6)$ are such that $\cos^{-1}\left(\frac{4}{21}\right) = \theta$, where θ is the angle between $\overrightarrow{x}$ and $\overrightarrow{y}$. Determine the value(s) of p.

9. Given vectors $\overrightarrow{r} = (1, 2, -1)$ and $\overrightarrow{s} = (-2, -4, 2)$, determine the components of a vector perpendicular to both $\overrightarrow{r}$ and $\overrightarrow{s}$. Show your work.

10. Find the value of p if the vectors of $\overrightarrow{r} = (p, p, 1)$ and $\overrightarrow{s} = (p, -2, -3)$ are perpendicular to each other.

11. (a) Given the vectors $\overrightarrow{p} = (-1, 3, 0)$ and $\overrightarrow{q} = (1, -5, 2)$ determine the components of a vector perpendicular to each of these vectors.

    (b) Given the vectors $\overrightarrow{m} = (1, 3, -4)$ and $\overrightarrow{n} = (-1, -2, 3)$ determine the components of a vector perpendicular to each of these vectors.

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