Linear Combinations of Vectors

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2021

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Suppose we have two non-collinear vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$. Then a *linear combination* of $\overrightarrow{u}$ and $\overrightarrow{v}$ is given by,

$$a\overrightarrow{u} + b\overrightarrow{v}$$

where $a, b, \in \mathbb{R}$.

A *spanning sest* is a set of vectors, $\{\overrightarrow{v_1}, \overrightarrow{v_2}\}$, say, for $\mathbb{R}^2$ is such that every vector in $\mathbb{R}^2$ can be written as a linear combination of $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$. The most common spanning set for $\mathbb{R}^2$ is $\{\hat{i}, \hat{j}\}$.

In $\mathbb{R}^3$ a *spanning set* $\{\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}\}$ where no two pairs of vectors of vectors of $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}$ are collinear, occurs when every vectors in $\mathbb{R}^3$ can be written as a linear combination of $\overrightarrow{v_1}, \overrightarrow{v_2}, \overrightarrow{v_3}$. The most common spanning set for $\mathbb{R}^3$ is $\{\hat{i}, \hat{j}, \hat{k}\}$.

Note: Any pair of non-collinear vectors $\overrightarrow{v_1}, \overrightarrow{v_2}$ in $\mathbb{R}^3$ spans a plane in $\mathbb{R}^3$.

Example

Show that $\overrightarrow{v} = (3, 5)$ can bet written as a linear combination of

(a) $\{(1, -4), (2, 3)\}$

(b) $\{(1, 0), (0, 1)\}$

Solution:

(a) Let's write $\overrightarrow{v}$ as a linear combination of $(1, -4)$ and $(2, 3)$ as,

$$\overrightarrow{v} = a(1, -4) + b(2, 3)$$

and see if we can find values for a and b. If we can, then $\overrightarrow{v}$ can be written as a linear combination of $(1, -4)$ and $(2, 3)$.

$$\begin{aligned} \overrightarrow{v} &= a(1, -4) + b(2, 3) \\ (3, 5) &= (a + 2b, -4 + 3b) \end{aligned}$$

From here we have the two equations,

$$\begin{aligned} 3 &= a + 2b & (1) \\ 5 &= -4a + 3b & (2) \end{aligned}$$

Multiplying equation (1) by 4 we get,

$$\begin{aligned} 12 &= 4a + 8b & (3) \\ 5 &= -4a + 3b & (4) \end{aligned}$$

Adding equations (3) and (4) we get,

$$\begin{aligned} 17 &= 11b \\ \therefore b &= \frac{17}{11} \end{aligned}$$

Plugging the value for b into equation 1 we get,

$$\begin{aligned} 3 &= a + 2b \\ 3 &= a + 2\left(\frac{17}{11}\right) \\ 33 &= 11a + 34 \\ \therefore 1 &= -\frac{1}{11} \end{aligned}$$

Since we were able to find values for a and b we have,

$$\overrightarrow{v} = -\frac{1}{1}(1, -4) + \frac{17}{11}(2, 3)$$

so $\overrightarrow{v}$ is a linear combination of $(1, -4)$ and $(2, 3)$.

(b) Let's do the same as in part (a).

$$\begin{aligned} \overrightarrow{v} = (3,5) &= a(1,0) + b(0,1) \\ (3,5) &= (a,b) \\ \therefore & \quad a = 3, \quad b = 5. \end{aligned}$$

Example

Show that $\{(1,3,-1),(2,0,1),(3,3,0)\}$ is not a spanning set for $\mathbb{R}^3$.

Solution: If we can show that one vectors in $\mathbb{R}^3$ cannot be written as a linear combination of $\{(1,3,-1),(2,0,1),(3,3,0)\}$ then we can conclude that $\{(1,3,-1),(2,0,1),(3,3,0)\}$ is not a spanning set for $\mathbb{R}^3$. Let's show that $(0,0,1)$ is not a linear combination of $\{(1,3,-1),(2,0,1),(3,3,0)\}$. Let's assume that $(0,0,1)$ can be written as a linear combination of $\{(1,3,-1),(2,0,1),(3,3,0)\}$ and show it's not possible.

$$(0,0,1) = a(1,3,-1) + b(2,0,1) + c(3,3,0), \quad \text{for } a,b,c \in \mathbb{R}.$$

From here we get the three equations,

$$0 = a + 2b + 3c \tag{5}$$
$$0 = 3a + 3c \tag{6}$$
$$1 = -a + b \tag{7}$$

From (2) we get,

$$\begin{aligned} 0 &= 3a + 3c \\ -3a &= 3c \\ \therefore a &= -c \end{aligned}$$

Plugging $a = -c$ into equation (1) we get,

$$\begin{aligned} 0 &= a + 2b + 3c \\ &= a + 2b - 3a \\ 0 &= -2c + 2b \\ 2c &= 2b \\ \therefore c &= b \end{aligned}$$

Plugging $c = b$ and $a = -c$ into equation (3) gives,

$$\begin{aligned} 1 &= -a + b \\ 1 &= -c + c \\ 1 &= 0 \end{aligned}$$

which is not possible. Therefore, $\{(1,3,-1),(2,0,1),(3,3,0)\}$ is not a spanning set for $\mathbb{R}^3$.

Example

Given two vectors $\overrightarrow{a} = (-1,-2,1)$ and $\overrightarrow{b} = (3,-1,1)$ does $\overrightarrow{c} = (-9,-4,1)$ lile on the plane determined by $\overrightarrow{a}$ and $\overrightarrow{b}$?

Solution: Is $\overrightarrow{c}$ in the span of $\overrightarrow{a}$ and $\overrightarrow{b}$? To determine this, we need to determine if there are values, m and n such that,

$$\begin{aligned} \overrightarrow{c} &= m\overrightarrow{a} + n\overrightarrow{b} \\ (-9,-4,1) &= m(-1,-2,1) + n(3,-1,1) \\ (-9,-4,1) &= (-m+3n, -2m-n, m+n) \end{aligned}$$

or,

$$\begin{aligned} -9 &= -m + 3n &\quad (8) \\ -4 &= -2m - n &\quad (9) \\ 1 &= m + n &\quad (10) \end{aligned}$$

From equation (8) we have,

$$\begin{aligned} 1-n &= m \\ -4 &= -2(1-n) \\ -4 &= -2+2n-n \\ -4 &= -2+n \\ \therefore -2 &= n \text{ and } m=3, \text{ substituting back into (8).} \end{aligned}$$

Thereofre,

$$\overrightarrow{c} = 3\overrightarrow{a} - 2\overrightarrow{b}$$

and $\overrightarrow{c}$ lies on the plane determined by $\overrightarrow{a}$ and $\overrightarrow{b}$.

Exercises

1. It is claimed that $\{(1,0,0),(0,1,0),(0,0,0)\}$ is a set of vectors spanning $\mathbb{R}^3$. Explain why it is not possible for these vectors to span $\mathbb{R}^3$.

2. In $\mathbb{R}^3$, the vector $\hat{i}=(1,0,0)$ spans a set. Describe the set spanned by this vector. Name two other vectors that would also span the same set.

3. It is proposed that the set $\{(0,0),(1,0)\}$ could be used to span $\mathbb{R}^2$. Explain why this is not possible.

4. Simplify each of the following linear combinations and write your answer in component form, $\overrightarrow{a}=\hat{i}-2\hat{j}$, $\overrightarrow{b}=\hat{j}-3\hat{k}$ and $\overrightarrow{c}=\hat{i}-3\ hatj+2\hat{k}$

   (a) $2(2\overrightarrow{a}-3\overrightarrow{b}+\overrightarrow{c})-4(-\overrightarrow{a}+\overrightarrow{b}-\overrightarrow{c})+(\overrightarrow{a}-\overrightarrow{c})$

   (b) $\frac{1}{2}(2\overrightarrow{a}-4\overrightarrow{b}-8\overrightarrow{c})-\frac{1}{3}(3\overrightarrow{a}-6\overrightarrow{b}+9\overrightarrow{c})$

5. (a) The set of vectors $\{(1,0,0),(0,1,0)\}$ spans a set in $\mathbb{R}63$. Describe this set.

   (b) Write the vector $(-2,4,0)$ as a linear combination of these vectors.

   (c) Explain why it is not possible to write $(3,5,8)$ as a linear combination of these vectors.

   (d) If the vector $(1,1,0)$ were added to this set, what would these three vector span in $\mathbb{R}^3$?

6. Solve for a,b and c in the following,

$$2(a,3,c)+3(c,7,c)=(5,b+c,15)$$

7. Determine the value for x such that the points A(-1, 3, 4), B(-2, 3, -1) and C(-5, 6, x) all lie on a plan that contains the origin.

8. The vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ span $\mathbb{R}^2$. For what values of m is it true that

$$(m^2 + 2m - 3)\overrightarrow{a} + (m^2 + m - 6)\overrightarrow{b} = \overrightarrow{0}?$$

Explain your reasoning.