Operations on vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

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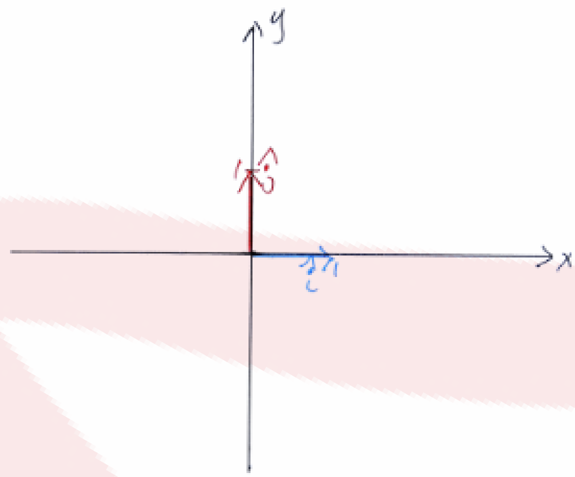
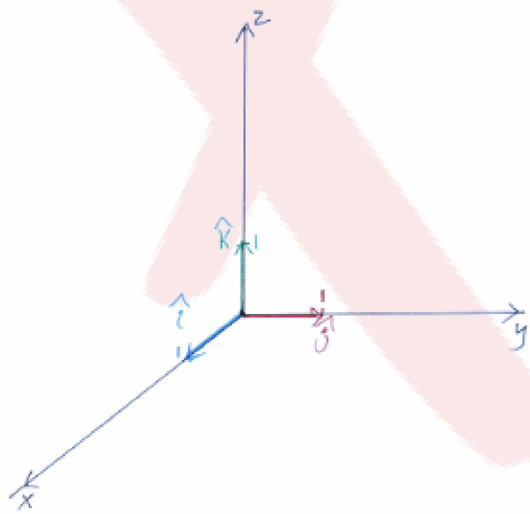
Let's start by defining the *unit vectors* in $\mathbb{R}^2$ and $\mathbb{R}^3$.
Unit vectors in $\mathbb{R}^2$:

$$\hat{i} = (1, 0),\ \hat{j} = (0, 1)$$



Unit vectors in $\mathbb{R}^3$:

$$\hat{i} = (1, 0, 0),\ \hat{j} = (0, 1, 0),\ \hat{k} = (0, 0, 1)$$



With these unit vectos we can represent vectors in $\mathbb{R}^2$ and $\mathbb{R}63$ in a couple of different ways.

(a) The first way is using unit vectors. For the point P(-3, 2) the position vector $\overrightarrow{OP}$ is $(-3, 2)$ which can be written as,

$$-3\hat{i} + 2\hat{j} = \overrightarrow{OP}$$

Similarly for point P(-1, 2, 4) in $\mathbb{R}^3$,

$$\overrightarrow{OP} = -\hat{i} + 2\hat{j} + 4\hat{k}$$

(b) The second way is to take a unit vector representation of a vector and rewrite in component for. For,

$$\overrightarrow{OA} - 02\hat{i} + 3\hat{j} = (-2, 3).$$

For,

$$\overrightarrow{OP} = 2\hat{i} - 3\hat{j} + 4\hat{k} = (2, -3, 4)$$

Adding two vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$

In $\mathbb{R}^2$: For two position vectors $\overrightarrow{a} = (a, b)$ and $\overrightarrow{c} = (c, d)$ in $\mathbb{R}^2$,

$$\overrightarrow{a} + \overrightarrow{c} = (a + c, b + d)$$

In $\mathbb{R}^3$: Similarly, in $\mathbb{R}^3$, for two position vectors $\overrightarrow{v} = (a, b, c)$, $\overrightarrow{u} = (x, y, z)$ in $\mathbb{R}^3$,

$$\overrightarrow{v} + \overrightarrow{u} = (a + x, b + y, c + z)$$

Scalar multiplication of a vector in $\mathbb{R}^2$ or $\mathbb{R}^3$

In $\mathbb{R}^2$: For $k \in \mathbb{R}$ and $\overrightarrow{v} = (a, b)$ in $\mathbb{R}^2$, $k\overrightarrow{v} = (ka, kb)$.

In $\mathbb{R}^3$: For $k \in \mathbb{R}$ and $\overrightarrow{v} = (a, b, c)$ in $\mathbb{R}^3$, $k\overrightarrow{v} = (ka, kb, kc)$.

Finding the position vector between two points A and B in $\mathbb{R}^2$ or $\mathbb{R}^3$

In $\mathbb{R}^2$: If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two poitns in $\mathbb{R}^2$ then the vector $\overrightarrow{AB}$ with head B and tail A is given by,

$$(x_2 - x_1, y_2 - y_1) = \overrightarrow{AB}$$

In $\mathbb{R}^3$: If $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are two points in $\mathbb{R}^3$ then the vector $\overrightarrow{AB}$ in $\mathbb{R}^3$ is given by,

$$(x_2 - x_1, y_2 - y_1, z_2 - z_1) = \overrightarrow{AB}$$

Magnitude of $\overrightarrow{AB}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$

In $\mathbb{R}^2$: For $\overrightarrow{AB} = (x_2 - x_1, y_2 - y_1)$ in $\mathbb{R}^2$ the magnitude is given by,

$$|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

In $\mathbb{R}^3$: For $\overrightarrow{AB} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$ in $\mathbb{R}^3$ the magnitude is given by,

$$|\overrightarrow{AB}| = \sqrt{(x_2 - x_2)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example

For point A(3, -1), B(-2, 1) and C(1, 1) find the following,

(a) write the position vector for points A and C using unit vectors.

(b) Write the position vectors for B ad C using component form.

(c) Find the vector $\overrightarrow{AC}$.

(d) Find the vector $3\overrightarrow{BC}$.

(e) Find the magnitude of the vector $\overrightarrow{AB}$.

Solution

(a)

$$\begin{aligned}\overrightarrow{OA} &= 3\hat{i} - \hat{j}\\ \overrightarrow{OC} &= \hat{i} + \hat{j}\end{aligned}$$

(b)

$$\begin{aligned}\overrightarrow{OB} &= (-2, 1)\\ \overrightarrow{OC} &= (1, 1)\end{aligned}$$

(c)

$$\overrightarrow{AC} = (1, 1) - (3, -1) = (1 - 3.1 - (-1)) = (-2, 2)$$

(d)

$$\overrightarrow{BC} = (-2, 1) - (1, 1) = (-2 - 1, 1 - 1) = (-3, 0)$$

Therefore, $3\overrightarrow{BC} = 3(-3, 0) = (-9, 0)$.

(e)

$$\begin{aligned}\overrightarrow{AB} &= (-2, 1) - (3, -1) = (-2 - 3, 1 - (-1)) = (-5, 2)\\ |\overrightarrow{AB}| &= \sqrt{(-5)^2 + 2^2}\\ &= \sqrt{25 + 4} = \sqrt{29}\end{aligned}$$

Exercises

1. For the vector $\overrightarrow{OA} = 3\hat{i} - 4\hat{j}$, calculate $|\overrightarrow{OA}|$.
2. (a) If $a\hat{i} + 5\hat{j} = (-3, b)$, determine the values of a and b.
   (b) Calculate $|(-3, b)|$ after finding b.
3. If $\overrightarrow{a} = (-60, 11)$ and $\overrightarrow{b} = (-40, -9)$, calculate each of the following,
   (a) $|\overrightarrow{a}|$ and $|\overrightarrow{b}|$
   (b) $|\overrightarrow{a} + \overrightarrow{b}|$ and $|\overrightarrow{a} - \overrightarrow{b}|$
4. Find a single vector equivalent to each of the following:
   (a) $2(-2, 3) + (2, 1)$
   (b) $-3(4, -9) - 9(2, 3)$
   (c) $-\frac{1}{2}(6, -2) + \frac{2}{3}(6, 15)$
5. Given $\overrightarrow{x} = 2\hat{i} - \hat{j}$ and $\overrightarrow{y} = -\hat{i} + 5\hat{j}$, find a vector equivalent to each of the following:
   (a) $3\overrightarrow{x} - \overrightarrow{y}$
   (b) $-(\overrightarrow{x} + 2\overrightarrow{y}) + 3(-\overrightarrow{x} - 3\overrightarrow{y})$
   (c) $2(\overrightarrow{x} + 3\overrightarrow{y}) - 3(\overrightarrow{y} + 5\overrightarrow{x})$
   (d) $|\overrightarrow{x} + \overrightarrow{y}|$
   (e) $|\overrightarrow{x} - \overrightarrow{y}|$
   (f) $|2\overrightarrow{x} - 3\overrightarrow{y}|$
   (g) $|3\overrightarrow{y} - 2\overrightarrow{x}|$
6. Parallelogram OBCD is determined by the vectors $\overrightarrow{OA} = (6, 3)$ and $\overrightarrow{OB} = (11, -6)$.

(a) Determine $\overrightarrow{OC}, \overrightarrow{BA}$ and $\overrightarrow{BC}$.

(b) Verify that $|\overrightarrow{OA}| = |\overrightarrow{BC}|$

7. ΔABC has vertices at A(2,3), B(6,6) and C(-4,11).

   (a) Sketch and label each of the points on a graph.

   (b) Calculate each of the lengths $|\overrightarrow{AB}|, |\overrightarrow{AC}|$ and $|\overrightarrow{CB}|$.

   (c) Verify that triangle ABC is a right triangle.

8. Determine the value of x and y in each of the following:

   (a) $3(x, 1) - 5(2, 3y) = (11, 33)$

   (b) $-2(x, x + y) - 3(6, y) = (6, 4)$

9. A(5,0) and B(0,2) are points on the x- and y- axes, respectively.

   (a) Find the coordinates of point P(a,0) on the x-axis such that $|\overrightarrow{PA}| = |\overrightarrow{PB}|$.

   (b) Find the coordinates of a point on the y-axis such that $|\overrightarrow{QB}| = |\overrightarrow{QA}|$.

10. Find the components of the unit vector in the direction opposite to $\overrightarrow{PQ}$ where $\overrightarrow{OP} = (11, 19)$ and $\overrightarrow{OQ} = (2, -21)$.