Systems of Equations
(Linear systems of equations)

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Systems of Equations

A *system of equations* or a linear system of equations or a linear system, is a group of set of linear equations. The goal when faced with a linear system of equations is usually to determine if these linear equations intersect or not. If they do intersect the linear system is said to be *consistent*. If they do not intersect then the linear system is said to be *inconsistent*. Once we have determined a linear system is consisten then the next step is to determine how many times the linear equations intersect. There are usually two possibilites, the linear equations intersect *uniquely*, a single point, or *infinitely* often, a line or plane. How do we solve linear systems? Good question. We use,

Elementary (row) operations

Elementary operations are applied to the linear systems. The goal is to create an *equivalent* linear system that is easier to solve. The reason for this is, two linear systems of equations are said to be *equivalent* if every solution of one linear system is also a solution of the second linear system. Elementary operations performe don a linear system is one way to create quivalent linear systme. So, what are these *elementary operations*? There are three elementary (row) operations.

1. **Multiplying** an equation by a non-zero constant (scalar).
2. **Interchanging** any pair of equations.
3. **Addition** of a non-zero multiple of one equation to second equation to replace the *second* equation.

Example

Solve the following system of linear equations,

$$\begin{aligned} 2x + y &= -9 \quad (1) \\ x + 2y &= -6 \quad (2) \end{aligned}$$

Solution: We'll start by trying to eliminate one of the variables from the linear system. Let's try and eliminate the variable x. To do this, let's first multiply equation (2) by 2 giving the *equivalent* system,

$$\begin{aligned} 2x + y &= -9 \quad (3) \\ 2x + 4y &= -12 \quad (4) \end{aligned}$$

Subtracting equation (4) from equation (3) we have,

$$\begin{aligned} -3y &= 3 \\ \therefore y &= -1 \end{aligned}$$

Plugging $y = -1$ into equation (2) we have,

$$\begin{aligned} x + 2(-1) &= -6 \\ x &= -6 + 2 \\ \therefore x &= -4 \end{aligned}$$

Therefore, (-4, -1) is the point of intersection of the two linear. This means the system is consistent with a unique solution.

Example

Find the solution, if any, of the linear system,

$$\begin{aligned} 2x + 3y - 5z &= -21 \quad (5) \\ x - 6y + 6z &= 8 \quad (6) \end{aligned}$$

Solution: Let's start by multiplying equation (6) by -2 and adding it to equation (5) to replace equation (5). This give the following equivalent system,

$$15y - 17z = -37 \quad (7)$$
$$x - 6y + 6z = 8 \quad (8)$$

Now we'll multiply equation (8) by 5/2 giving the equivalent system,

$$15y - 17z = -37 \quad (9)$$
$$\frac{5}{2}x - 15y + 15z = 20 \quad (10)$$

We'll add equation (9) to equation (10) replacing equation (10) giving the equivalent system,

$$15y - 17z = -37 \quad (11)$$
$$\frac{5}{2}x - 2z = -17 \quad (12)$$

Notice that from equation (11) we write y in terms of z and from equation (12) x in terms of z as well,

$$y = -\frac{37}{15} + \frac{17}{15}z$$
$$x = -\frac{34}{5} + \frac{4}{5}z$$

If we let $z = t \in \mathbb{R}$ then we have the solution,

$$\begin{cases} x = -\frac{34}{5} + \frac{4}{5}t \\ y = -\frac{34}{5} + \frac{4}{5}t \\ z = t \end{cases}$$

Since we have a solution, the system is consistent. Since $z = t \in \mathbb{R}$, we have infinitely many solutions. Notice that the solution has the form of a line in parametric form.

Exercises

1. Determine whether $x = -7, y = 5$ and $z = 3/4$ is a solution to the following system of equations,

$$\begin{aligned} x - 3y + 4z &= -19 \\ x - 8z &= -13 \\ x + 2y &= 3 \end{aligned}$$

2. Solve each system of equations, and state whether the systems given below are equivalent or not. Justify your answer.

   (a)

$$\begin{aligned} x &= -2 \\ 3y &= -9 \end{aligned}$$

   (b)

$$\begin{aligned} 3x + 5y &= -21 \\ \frac{1}{6}x - \frac{1}{2}y &= \frac{7}{6} \end{aligned}$$

3. Solve each of the following systems using elementary operations,

   (a)

$$\begin{aligned} 2x - y &= 11 \\ x + 5y &= 11 \end{aligned}$$

   (b)

$$\begin{aligned} 2x + 5y &= 19 \\ 3x + 4y &= 11 \end{aligned}$$

(c)

$$\begin{aligned} -x + 2y &= 10 \\ 3x + 5y &= 3 \end{aligned}$$

4. Write a solution to each equation using parameters.
   (a) $2x - y = 3$
   (b) $x - 2y + z = 0$

5. Given the system of equations,

$$\begin{aligned} x + y &= 6 \\ 2x + 2y &= k \end{aligned}$$

   determine the value(s) of the constant k for which the following system of equations has,
   (a) no solutions
   (b) one solution
   (c) infinitely many solutions

6. (a) Solve the following system of equations for x and y,

$$\begin{aligned} x + 3y &= a \\ 2x + 3y &= b \end{aligned}$$

   (b) Explain why this system of equations will always be consistent regardless of the values of a and b.

7. Solve each of the equations using elementary operations.
   (a)

$$\begin{aligned} x + y + z &= 0 \\ x - y &= 1 \\ y - z &= -5 \end{aligned}$$

(b)

$$\begin{aligned} 2x - 3y + z &= 6 \\ x + y + 2z &= 31 \\ x - 2y - z &= -17 \end{aligned}$$

(c)

$$\begin{aligned} 2x - 7 &= 0 \\ 2y - z &= 7 \\ 2z - x &= 0 \end{aligned}$$

8. Consider the following system of questions,

$$\begin{aligned} x + 2y &= -1 \\ 2x + k^2 y &= k \end{aligned}$$

Determine the values of k for which this system of equations has,

(a) no solutions

(b) an infinite number of solutions

(c) a unique solution

