The distance from a point to a plane

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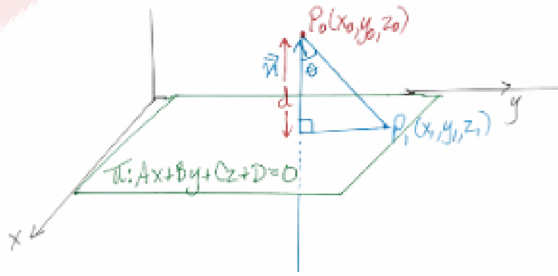


Let $P_0(x_0, y_0, z_0)$ be a point not on the plane $\pi : Ax + By + Cz + D = 0$ and $P_1(x_1, y_1, z_1)$ a point on the plane π. The normal to the plane is given by $\overrightarrow{n} = (A, B, C)$. Let θ be the angle between the normal $\overrightarrow{n}$ and $\overrightarrow{P_0P_1}$. The distance between the point P_0 and the plane π is given by the projection of $\overrightarrow{P_0P_1}$ onto the normal $\overrightarrow{n}$ or,

$$\begin{aligned}
|\overrightarrow{P_0P_1}|\cos\theta &= d \\
|\overrightarrow{n}||\overrightarrow{P_0P_1}|\cos\theta &= |\overrightarrow{n} \cdot \overrightarrow{P_0P_1}| \\
d &= \frac{|\overrightarrow{n} \cdot \overrightarrow{P_0P_1}|}{|\overrightarrow{n}|} \\
d &= \frac{(A, B, C) \cdot (x_0 - x_1, y_0 - y_1, z_0 - z_1)}{\sqrt{A^2 + B^2 + C^2}} \\
d &= \frac{|A(x_0 - x_1) + B(y_0 - y_1) + C(z_0 - z_1)|}{\sqrt{A^2 + B^2 + C^2}} \\
d &= \frac{|Ax_0 - Ax_1) + By_0 - By_1 + Cz_0 - Cz_1|}{\sqrt{A^2 + B^2 + C^2}} \\
d &= \frac{|Ax_0 + By_0 + Cz_0 - Ax_1 - By_1 - Cz_1|}{\sqrt{A^2 + B^2 + C^2}}
\end{aligned}$$

Therefore, the distance from a point $P_0(x_0, y_0, z_0$ to a plane $\pi : Ax +$

$By + Cz + D = 0$ is given by,

$$\boxed{d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}}$$

Example

Find the distance between the two planes $\pi_1 : 2x - y + 2z + 4 = 0$ and $\pi_2 : 2x - y + 2z + 16 = 0$.

Solution: we need to find a point P_0 on one plane and then find the distance from this point P_0 to the other plane. Let's find a point on plane π_1. Let's choose $P_0(1, 4 - 1)$ and let's check that P_0 indeed lies on plane π_1.

$$2(1) - 4 + 2(-1) + 4 = 2 = 4 = 2 + 4 = 0$$

Therefore, $P_0(1, 4, -1)$ lies on the plane π_1. Now we can find the distance from P_0 to plane π_2 using the formula.

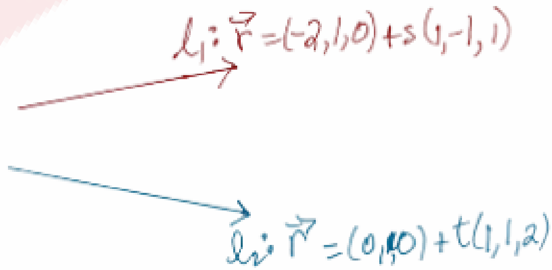
$$\begin{aligned} d &= \frac{|2(1) - 4 + 2(-1) + 16|}{\sqrt{4 + 1 + 4}} \\ &= \frac{|2 - 4 - 2 + 16|}{\sqrt{9}} \\ &= \frac{12}{3} \\ \therefore d &= 4 \end{aligned}$$

Therefore, the distance between the two planes is 4.

Example

What is the distance between $l_1 : \overrightarrow{r} = (-2, 1, 0) + s(1, -1, 1)$ and $l_2 : \overrightarrow{r} = (0, 1, 0) + t(1, 1, 2)$, where $t, s, \in \mathbb{R}$.

Solution:



If we can find a plane π_2 containing say line l_2 parallel to line l_1 then we can take a point on line l_1 and find its distance to plane π_2. How do we find a plane π_2 parallel to l_1? To write the equation of a plane we need two direction vectors and a point that lies on the plane. Since we want our plane π_2 to contain line l_2 and be parallel to line l_1 let's take the direction vectors for plane π_2 from lines l_1 and l_2, so $\overrightarrow{m_1}$ and $\overrightarrow{m_2}$, respectively. Using these two vectors we can find the vector perpendicular, $\overrightarrow{n}$, which will be the normal to the plane π_2. To find $\overrightarrow{n}$ we need to take the cross produce of $\overrightarrow{m_1}$ and $\overrightarrow{m_2}$.

$$
\begin{aligned}
\overrightarrow{n} &= \overrightarrow{m_1} \times \overrightarrow{m_2} \\
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 1 & 1 & 2 \end{vmatrix} \\
&= \hat{i}(-2-1) - \hat{j}(2-1) + \hat{k}(1+1) \\
&= -3\hat{i} - \hat{j} + 2\hat{k}
\end{aligned}
$$

Therefore, the normal to plane π_2 is $\overrightarrow{n} = (-3, -1, 2)$. We also want plane π_2 to include l_2. So we'll take a point from l_2 to lie on π_2. Let's take the point P(0, 1, 0) to lie on the plane π_2. Putting this all together we have,

$$\begin{aligned} -3x - y + 2z + D &= 0 \\ -3(0) - 1 + 2(0) + D &= 0 \\ -1 + D &= 0 \\ D &= 1. \end{aligned}$$

Therefore, our second plane is given by, $\pi_2 : -3x - y + 2z + 1 = 0$. Now that we have our plane, we'll choose a point on l_1. Let's choose $P_0(-2, 1, 0)$. Now we can use the formula to find the distance between P_0 and π_2.

$$\begin{aligned} d &= \frac{|-3(-2) - 1 + 2(0) + 1|}{\sqrt{9 + 1 + 4}} \\ &= \frac{|6 - 1 + 0 + 1|}{\sqrt{14}} \\ &= \frac{6}{\sqrt{14}} \end{aligned}$$

Therefore, the distance between l_1 and l_2 in $\mathbb{R}^3$ is $\frac{6}{\sqrt{14}}$.

Exercises

1. Determine the following distances,
   (a) the distance from A(3, 1, 0) to the plane with the equation 20x - 4y + 5z +7 =0.
   (b) The distance from B(0, -1, 0) to the plane with equation 2x + y + 2z -8 =0.
   (c) the distance from C(5, 1, 4) to the plane with equation 3x -4y -1 =0.
   (d) the distance from D(1, 0, 0) to the plane with equation 5x -12 y =0.
   (e) the distance from E(-1, 0, 1) to the plane with equation 18x -9y + 18z -11 =0.
2. For the planes $\pi_1 : 3x+4y-12z-26 = 0$ and $\pi_2 : 3x+4y-12z+39 = 0$, determine
   (a) the distance between π_1 and π_2
   (b) an equation for a plane midway between π_1 and π_2
   (c) the coordinate of a point that is equidistant from π_1 and π_2.
3. Determine the following distances,
   (a) the distance from P(1, 1, -3) to the plane with equation y+3=0
   (b) the distance from Q(-1, 1, 4) to the plane with equation x-3=0
   (c) the distance from R(1, 0, 1) to the plane with equation z+1 =0
4. Points A(1, 2, 3), B(-3, -1, 2) and C(13, 4, -1) lie on the same plane. Determine the distance from P(1, -1, 1) to the plane containing these three points.

5. The distance from R(3, -3, 1) to the plane with equation Ax + 2y + 6z =0 is 3. Determine all possible value(s) of A for which this is true.

6. Determine the distance between the lines $\overrightarrow{r} = (0, 1, -1) + s(3, 0, 1),\ s \in \mathbb{R}$, and $\overrightarrow{r} = (0, 0, 1) + t(1, 1, 0),\ t \in \mathbb{R}$.