Distance from a point to a line in $\mathbb{R}^2$ and $\mathbb{R}^3$

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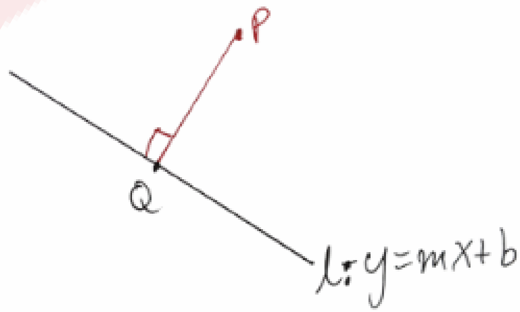
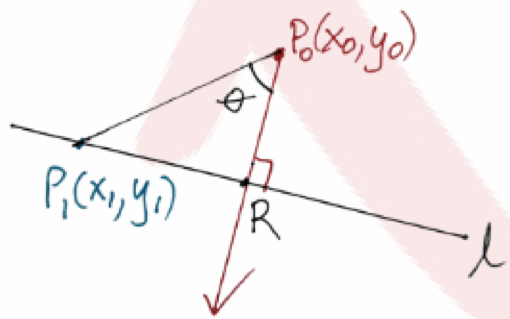


$l : y = mx + b$

Recall how we would have to find the distance between a point and a line l in $\mathbb{R}^2$. We would have to,

1. Find the equation of the line l_1 through P and perpendicular to the given line l. Recall that the slope of the line l_1, perpendicular to l, is $-\frac{1}{m}$.
2. Find the point of intersection Q of lines l_1 and l.
3. Find the distance between the points P and Q.

In $\mathbb{R}^2$



Let $P_0(x_0, y_0)$ and $P_1(x_1, y_1)$. We will use the Cartesian equation of the line l given by $l : Ax + By + C = 0$. In the above diagram d is the scalar project of $\overrightarrow{P_1P_0}$ onto PR. Letting θ be the angle between $\overrightarrow{P_1P_0}$ and PR we have,

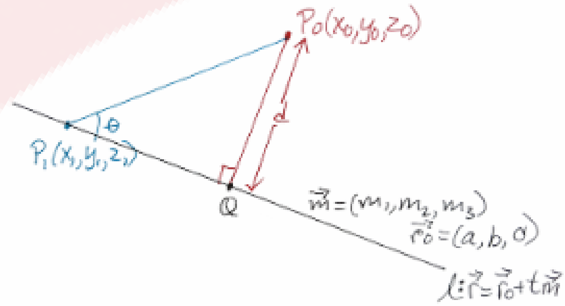
$$\begin{aligned}
d &= |\overrightarrow{P_1P_0}| \cos\theta \\
|\overrightarrow{P_1P_0} \cdot \overrightarrow{n}| &= |\overrightarrow{P_1P_0}||\overrightarrow{n}| \cos\theta \\
\frac{|\overrightarrow{P_1P_0} \cdot \overrightarrow{n}|}{|\overrightarrow{n}|} &= |\overrightarrow{P_1P_0}| \cos\theta = d \\
\frac{|(x_1 - x_0, y_1 - y_0) \cdot (A, B)|}{\sqrt{A^2 + B^2}} &= d \\
\frac{|A(x_1 - x_0) + B(y_1 - y_0)|}{\sqrt{A^2 + B^2}} &= d \\
\frac{|Ax_1 - Ax_0 + By_1 - By_0|}{\sqrt{A^2 + B^2}} &= d \\
\frac{|Ax_1 + By_1 - Ax_0 - By_0|}{\sqrt{A^2 + B^2}} &= d \\
\frac{|-C - Ax_0 - By_0|}{\sqrt{A^2 + B^2}} &= d
\end{aligned}$$

Therefore, the distance from a point to a line in $\mathbb{R}^2$ is given by,

$$\boxed{d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}}$$

In $\mathbb{R}^3$

Let's consider the same situation in $\mathbb{R}^3$.



Let $P_0(x_0, y_0, z_0)$ and $P_1(x_1, y_1, z_1)$. Let's consider the vector question of the line $l : \overrightarrow{r} = \overrightarrow{r_0} + t\overrightarrow{m}$, where $\overrightarrow{r_0} = (a, b, c)$ and direction vector $\overrightarrow{m} = (m_1, m_2, m_3)$. The distance d betwen $P - 0$ and the line l is $d = |\overrightarrow{P_1P_0}| \sin\theta$, where θ is the angle between $\overrightarrow{P_1P_0}$ and PQ. Recall the cross product,

$$
\begin{aligned}
d &= |\overrightarrow{P_1P_0}| \sin\theta \\
|\overrightarrow{m} \times \overrightarrow{P_1P_0}| &= |\overrightarrow{m}||\overrightarrow{P_1P_0}| \sin\theta \\
\frac{|\overrightarrow{m} \times \overrightarrow{P_1P_0}|}{|\overrightarrow{m}|} &= |\overrightarrow{P_1P_0}| \sin\theta = d
\end{aligned}
$$

Therefore, the distance from a point to a line in $\mathbb{R}^3$ is given by,

$$
\boxed{d = \frac{|\overrightarrow{m} \times \overrightarrow{P_1P_0}|}{|\overrightarrow{m}|}}
$$

Exercises

1. Determine the distance from P(-4, 5) to each of the following lines,
   (a) $3x + 4y - 5 = 0$
   (b) $5x - 12y + 24 = 0$
   (c) $9x - 40y = 0$
2. Determine the distance between the following parallel lines,
   (a) $2x - y + 1 = 0,\ 2x - y + 6 = 0$
   (b) $7x - 24y + 168 = 0,\ 7x - 24y - 336 = 0$
3. Determine the distance from P(-2, 3) to each of the following lines,
   (a) $\overrightarrow{r} = (-1, 2) + s(3, 4),\ s \in \mathbb{R}$
   (b) $\overrightarrow{r} = (1, 0) + t(5, 12),\ t \in \mathbb{R}$
   (c) $\overrightarrow{r} = (1, 3) + p(7, -24),\ p \in \mathbb{R}$
4. Calculate the distance between the following lines,
   (a)
   $$\frac{x-1}{4} = \frac{y}{-3},\quad \frac{x}{4} = \frac{y+1}{-3}$$
   (b)
   $$5x + 12y = 120,\ 5x + 12y + 120 = 0$$
5. Calculate the distance between point P and the given line.
   (a) P(1, 2, -1); $\overrightarrow{r} = (1, 0, 0) + s(2, -1, 2),\ s \in \mathbb{R}$
   (b) P(0, -1, 0); $\overrightarrow{r} = (2, 1, 0) + t(-4, 5, 20),\ t \in \mathbb{R}$
   (c) P(2, 3, 1); $\overrightarrow{r} = p(12, -3, 4),\ p \in \mathbb{R}$

6. (a) Determine the coorindates of the point on the line $\rightarrow r = (1, -1, 2) + s(1, 3, -1),\ s \in \mathbb{R}$, that produces the shortest distance between the line and a point with coordinates (2, 1, 3).
   (b) What is the distance between the given point and the line?
7. Two planes with equations $x - y + 2z = 2$ and $x + y - z = -2$ intersect along line l. Determine the distance from P(-1, 2, -1) to l, and determine the coordinates of the point on l that vies the minimal distance.

