Intersection of a Line and Plane

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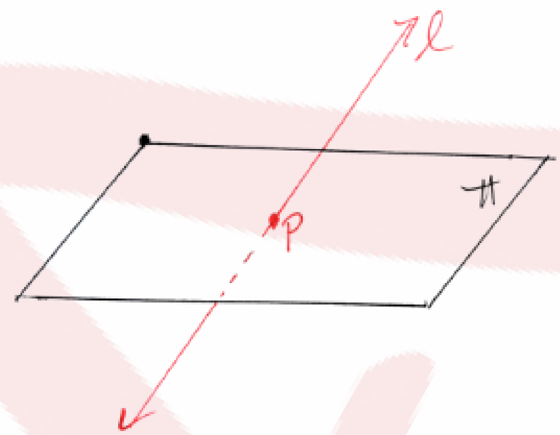
Two planes in $\mathbb{R}^3$

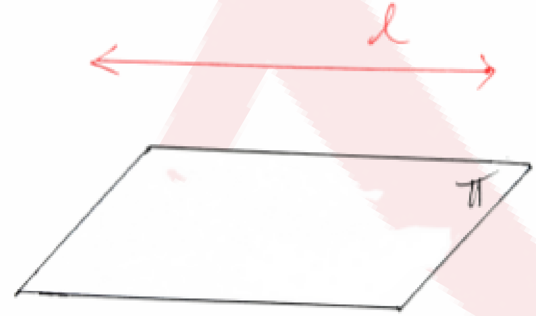
Let's what the possibilities of a plane and line are in $\mathbb{R}^3$, visually.

Case 1: Line and Plane intersect



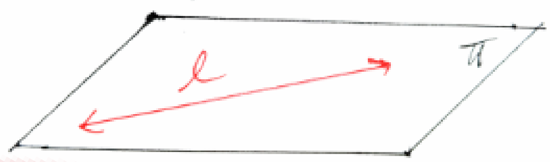
The line l intersects the plane π in one point P.

Case 2: Line and Plane are parallel



The line l is parallel to the plane π. This means they have a common direction vector or the the direction vectors of the line and plane are multiples of each other.

Case 3: Line lies on Plane

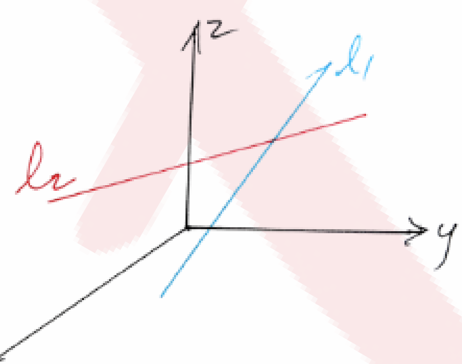


The line l lies entirely on the plane π.

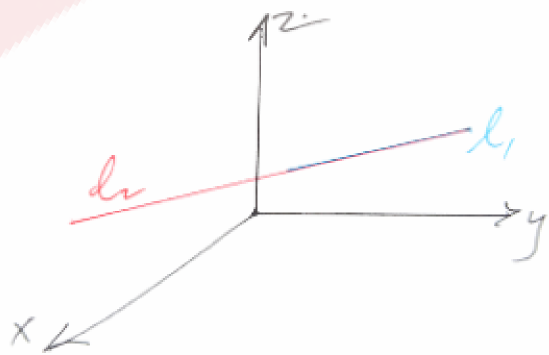
Two lines in $\mathbb{R}^3$

Now, let's consider two lines in $\mathbb{R}^3$.

Intersect in one point

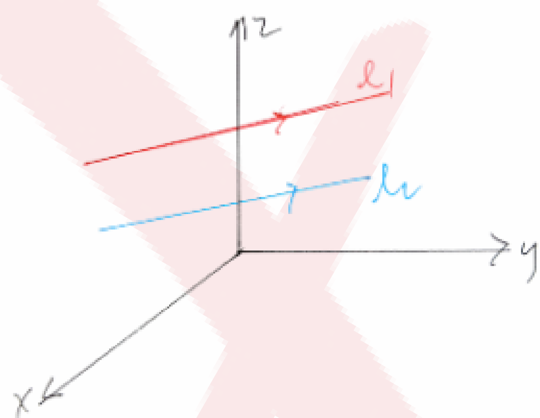


Intersect in a line



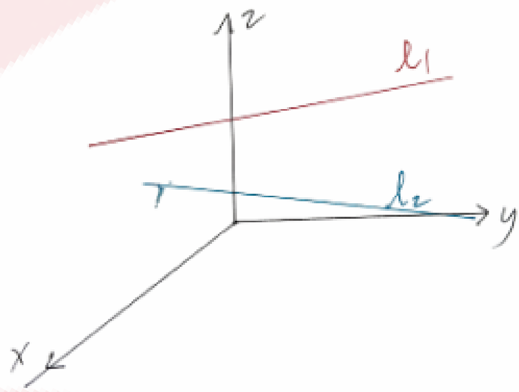
When two lines l_1 and l_2 in $\mathbb{R}^3$ intersect in a line, the two lines are *coincident*, that is, they are the same line, $l_1 = l_2$.

Parallel line



Two lines, l_1 and l_2, in $\mathbb{R}^3$ are parallel, then they have the same direction vector but have no common point that lies on them.

Skew lines

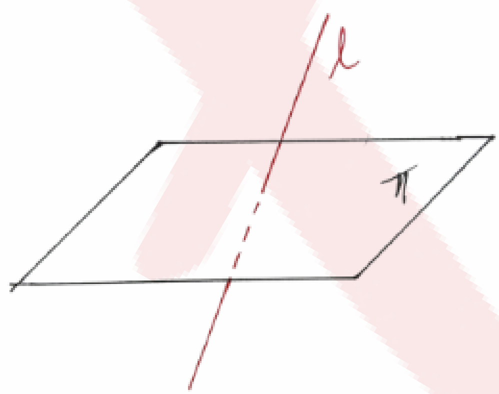


Two lines, l_1 and l_2, in $\mathbb{R}^3$ are skew when they not only do not share any points in common but also do not have the same direction vector.

Example

Determine the points of intersection of the line l and the plane π where,

$$\begin{aligned} l: &\quad \overrightarrow{r} = (3,1,2) + s(1,-4,-8),\ s \in \mathbb{R} \\ \pi: &\quad 4x + 2y - z - 8 = 0 \end{aligned}$$



Solution: We need to insert the line into the plane but first, let's rewrite the line in parametric form to make our life easier. The parametric form of the line l is given by,

$$\begin{aligned} x &= 3+s \\ y &= 1-4s \\ z &= 2-8s, \; s \in \mathbb{R} \end{aligned}$$

Plugging the point into the plane we get,

$$\begin{aligned} 4x+2y-z-8 &= 0 \\ 4(3+s)+2(1-4s)-(2-8s) &= 0 \\ 12+4s+2-8s-2+8s-8 &= 0 \\ 4+4s &= 0 \\ s &= -1 \end{aligned}$$

Using this value for s, we can use it to determine the point of intersection but plugging $s=-1$ into the parametric equation for the line. Doing this we get,

$$\begin{aligned} x &= 3-1=2 \\ y &= 1-4(-1)=5 \\ z &= 2-8(-1)=10 \end{aligned}$$

Therefore, the point of intersection of the plane π and line l is (2, 5, 10).

Example

What is the point of intersection of line,

$$l : x=2+t, y2+2t, z=9+8t$$

and plane

$$\pi : 2x-5y_z-6=0$$

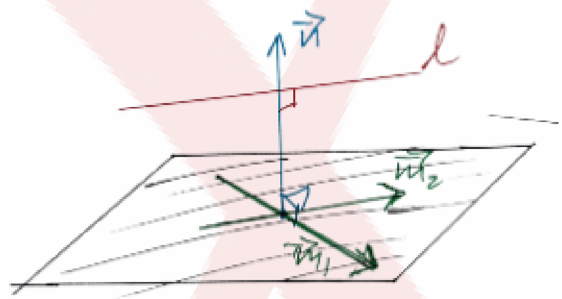
Solution: We will plug the line l into the plane π and solve for t and see what we get.

$$\begin{aligned} 2(2+t)-5(2+2t)+(9+8t)-6 &= 0 \\ 4+2t-10-10t+9+8t &= 0 \\ -3+0t &= 0 \\ -3 &= 0 \end{aligned}$$

which is not possible. This means tha l and π do **not** intersect.

Note: Another way to answer this question is to consider the direction vector of l, $\overrightarrow{m}=(1,2,8)$ and the normal of the plane π, $\overrightarrow{n}=(2,-5,1)$. We see that they are actually perpendicular,

$$\begin{aligned} \overrightarrow{m}\cdot\overrightarrow{n} &= (1,2,8)\cdot(2,-5,1) \\ &= 2-10+8 \\ &= 0 \end{aligned}$$



Therefore, the direction vectors of the plane π are parallel to the direction vector of the line l, thus the line l and plane π are parallel and have no point of intersection.

Example

Determine the point(s) of intersection of the two lines,

$$\begin{aligned} l_1 : \overrightarrow{r} &= (-3,1,4) + s(-1,1,4),\ s \in \mathbb{R} \\ l_2 : \overrightarrow{r} &= (1,4,6) + t(-6,-1,6),\ t \in \mathbb{R} \end{aligned}$$

Solution: First we note that the lines are not parallel because $\overrightarrow{m_1} = (-1,1,4) \neq (-6,-1,6) = \overrightarrow{m_2}$ Let's rewrite each line in parametric form and then equate coorindates and solve for s and t.

$$\begin{aligned} l_1 :\ x &= -3 - s & (1) \\ y &= 1 + s & (2) \\ z &= 4 + 4s & (3) \end{aligned}$$

and,

$$\begin{aligned} l_2 :\ x &= 1 - 6t & (4) \\ y &= 4 - t & (5) \\ z &= 6 + 6t & (6) \end{aligned}$$

We now equate equations corresponding equations (1), (2), (3) to (4), (5), (6), respectively. This gives,

$$\begin{aligned} -3 - s &= 1 - 6t \\ 1 + s &= 4 - t \\ 4 + 4s &= 6 + 6t \end{aligned}$$

Simplifying we have,

$$\begin{aligned} -4 - s + 6t &= 0 & (7) \\ -3 + s + t &= 0 & (8) \\ -2 + 4s - 6t &= 0 & (9) \end{aligned}$$

Adding equations (7) and (8) we get,

$$\begin{aligned} -7 + 7t &= 0 \\ \therefore t &= 1 \end{aligned}$$

Plugging $t = 1$ into equation (9) we have,

$$\begin{aligned} -2 + 4s - 6(1) &= 0 \\ 4s &= 6 + 2 \\ 4s &= 8 \\ \therefore s &= 2 \end{aligned}$$

Plugging $t = 1$ and $s = 2$ into the parametric equations for l_1 (or l_2) we have,

$$\begin{aligned} x &= -3 - 2 = -5 \\ y &= 1 + 2 = 3 \\ z &= 4 + 4(2) = 12 \end{aligned}$$

Therefore, (-5, 3, 12) is the point of intersection of lines l_1 and l_2.

Example

Determine the point of intersection of l_1 and l_2 where,

$$\begin{aligned} l_1 : \ x &- -1 + s \\ y &= 3 + 4s \\ z &= 6 + 5s \end{aligned}$$

and,

$$\begin{aligned} l_2 : \ x &= 4 - t \\ y &= 17 + 2t \\ z &= 30 - 5t \end{aligned}$$

Solution: Equate coordinates for l_1 and l_2 and solve for t and s. Hopefully things work out!

$$\begin{aligned} -1+s &= 4-t \\ 3+4s &= 17+2t \\ 6+5s &= 30-5t \end{aligned}$$

Simplifying the above we have,

$$-5+s+t = 0 \quad (10)$$
$$-14+4s-2t = 0 \quad (11)$$
$$-24+5s+5t = 0 \quad (12)$$

$4\times$ equation (10) - equation (11) we have,

$$\begin{aligned} 6+6t &= 0 \\ \therefore t &= -1 \end{aligned}$$

Plugging $t=-1$ into equation (12) gives,

$$\begin{aligned} -24+5s+5(1) &= 0 \\ -24+5s-5 &= 0 \\ -29+5s &= 0 \\ \therefore s &= \frac{29}{5} \end{aligned}$$

Plugging $t=-1$ and $s=29/5$ into l_2 we have,

$$\begin{aligned} x &= 5 \\ y &= 15 \\ z &= 35 \end{aligned}$$

and plugging $t = -1$ and $s = 29/5$ into l_1 we have,

$$
\begin{aligned}
x &= -1 + \frac{29}{5} = \frac{24}{5} \\
y &= 3 + 4\left(\frac{29}{5}\right) = \frac{15 + 4(29)}{5} \\
z &= 6 + 5\left(\frac{29}{5}\right) = 35
\end{aligned}
$$

Since the x-coordinates on l_1 and l_2 when $t = -1$ and $s = 29/5$ are not equal, the lines l_1 and l_2 do not intersect.

Exercises

1. (a) If a line and a plane intersect, in how many diofferent ways casn this occur? Describe each case.

   (b) It is only possible to have zero, one or an infinite number of intersections between a line and a plane. Explain why it is not possible to have a finite number of intersections, other than zero or one, between a line and a plane.

2. A line has the equation $\overrightarrow{r} = s(1, 0, 0),\ s \in \mathbb{R}$, and a plane has the equation $y = 1$.

   (a) Describe the line.

   (b) Describe the plane.

   (c) Sketch the line and the plane.

   (d) Describe the nature of the intersection between the line and the plane.

3. Show that the line, $l : x = -2 + t, y = 1 - t, z = 2 + 3t,\ t \in \mathbb{R}$ lies on the plane $\pi : x + 4y + z - 4 = 0$.

4. Show that the line $l : x = 1 + 2t, y = -2 + 5t, z = 1 + 4t,\ t \in \mathbb{R}$ and plane $\pi : 2x - 4y + 4z - 13 = 0$ do not intersect.

5. Determine the point(s) of intersection between the following line and plane, if any.

$$l : \frac{x-1}{4} = \frac{y+2}{-1} = z - 3$$

and

$$\pi : 2x + 7y + z + 15 = 0.$$

6. Determine the point(s) of intersection, if any, between lines l_1 and l_2 given below,

$$l_1 : \overrightarrow{r} = (3, 1, 5) + s(4, -1, 2),\ s \in \mathbb{R}$$

and

$$l_2 : x = 4 + 13t, y = 1 - 5t, z = 5t,\ t \in \mathbb{R}$$

7. Which of the following pairs of lines are skew. Justify your answers.

(a)

$$\begin{aligned} \overrightarrow{r} &= (-2, 3, 4) + p(6, -2, 3),\ p \in \mathbb{R} \\ \overrightarrow{r} &= (-2, 3, -4) + q(6, -2, 11),\ q \in \mathbb{R} \end{aligned}$$

(b)

$$\begin{aligned} \overrightarrow{r} &= (4, 1, 6) + t(1, 0, 4),\ t \in \mathbb{R} \\ \overrightarrow{r} &= (2, 1, -8) + s(1, 0, 5),\ s \in \mathbb{R} \end{aligned}$$

(c)

$$\begin{aligned} \overrightarrow{r} &= (2, 2, 1) + m(1, 1, 1),\ m \in \mathbb{R} \\ \overrightarrow{r} &= (-2, 2, 1) + p(3, -1, -1),\ p \in \mathbb{R} \end{aligned}$$

(d)

$$\begin{aligned} \overrightarrow{r} &= (9, 1, 2) + m(5, 0, 4),\ m \in \mathbb{R} \\ \overrightarrow{r} &= (8, 2, 3) + s(4, 1, -2),\ s \in \mathbb{R} \end{aligned}$$

8. The line with the equation $\overrightarrow{r} = (-3, 2, 1) + s(3, -2, 7,\ s \in \mathbb{R}$, intersects the z-axis at the point Q(0, 0, q). Determine the value of q.

9. (a) Show that the lines $l_1 : \overrightarrow{r} = (-2, 3, 4) + s(7, -2, 3),\ s \in \mathbb{R}$, and $l_2 : \overrightarrow{r} = (-30, 11, -4) + t(7, -2, 2),\ t \in \mathbb{R}$, are coincident by writing each line in parametric form and comparing components.

   (b) Show that (-2, 3, 4) lies on line l_2. How does this show that the lines are coincident.

10. The lines $\overrightarrow{R} = (-3, 8, 1) + s(1, -1, 1),\ s \in \mathbb{R}$, and $\overrightarrow{R} = (1, 4, 2) + t(-3, k, 8),\ t \in \mathbb{R}$ intersect in a point.

    (a) Determine the value of k.

    (b) What are the coordinate of the point of intersection?

11. The lines $\overrightarrow{r} = (-1, 3, 2) + p(5, -2, 10),\ p \in \mathbb{R}$ and $\overrightarrow{r} = (4, -1, 1) + t(0, 2, 11),\ t \in \mathbb{R}$ intersect at point A.

    (a) Determine the coordinate of point A.

    (b) Determine the vector equation for th line that is perpendicular to the two given lines and passes through point A.

12. (a) Show that the lines,

$$\frac{x}{1} = \frac{y-7}{-8} = \frac{z-1}{2} \text{ and } \frac{x-4}{3} = \frac{z-1}{-2}, y = -1$$

    lie on the plane with equation $2x + y + 3z = 10 = 0$.

    (b) Determine the point of intersection of these two lines.