Matrices

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Matrices (Introduction)

Considering a linear system of equations,

$$\begin{aligned} x + 2y + 2z &= 9 \\ x + y &= 1 \\ 2x + 3y - z &= 1 \end{aligned}$$

a *matrix* is a box of numbers. For a linear system we have a couple of matrices of interest. The *coefficient matrix* of a linear system, in particular for the linear system above, is given by,

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 0 \\ 2 & 3 & -1 \end{bmatrix}$$

contains all the coefficients for the variables x, y and z in the linea system. Notice the first equation's coefficients corresponds to the first row of the matrix, and so on. The *augmented matrix* for the linear system is given by,

$$\begin{bmatrix} 1 & 2 & 2 & 9 \\ 1 & 1 & 0 & 1 \\ 2 & 3 & -1 & 1 \end{bmatrix}$$

Notice that the augmented matrix looks like the coefficient matrix with one extra right most column consisting of the values to the right of the equal signes for the equations in the linear system. Notice that in both matrices the columns correpond to a particular variable in the linear system. That is, the first column of the matrix corresponds to all the coefficient for the variable x and so on. This means that the equations in the linear system shoudl be written with all the variables in the same order.

Recall the elementary operations on a linear system. These elementary operations have equivlaent *elementary row operations on matrices.*

Elementary row operations for matrices

1. **Multipy** a row by a non-zero constant.
2. **Interchange** any pair of rows.
3. **Add** a multiple of one row to a second row to replace the second row.

Note: A row in the matrix represents an equation in the linear system.

Elementary row operations may be performed on a matrix. Any matrix that results from elementary row operations is *row equivalent* to the original matrix. Suppose elementary row operations are performed on a matrix until the following matrix structure occurs,

$$\begin{bmatrix} \# & \# & \# & \# \\ 0 & \# & \# & \# \\ 0 & 0 & \# & \# \end{bmatrix}$$

where # represents any non-zero number. This final structure is called *row echelon form.*

Properties of a matrix in row-echelon form

1. All rows that are all zeros are moved to the bottom of the matrix.
2. The first non-zero entry in a row, called the *leading entry*, must contain zeros below it.

We can take the row-echelon form a reduce it further to the following structure,

$$\begin{bmatrix} 1 & 0 & 0 & \# \\ 0 & 1 & 0 & \# \\ 0 & 0 & 1 & \# \end{bmatrix}$$

This form is called *reduced row-echelon form.*

Properties of a matrix in reduced row-echelon form

1. The matrix is in row-echelon form.
2. The first non=zero entry in each row is a 1. This 1 is called a *leading 1*.
3. Any column containing a leading 1 has all other column entries equal to 0.

Example

Given the following linear system,

$$\begin{aligned} x + 2y + 2z &= 9 \\ x + y &= 1 \\ 2x + 3y - z &= 1 \end{aligned}$$

(a) Find the coefficent matrix for the linear system.

(b) Find the augmented matrix.

(c) Use elementary row operations to find the row-echelon form of the augmented matrix.

(d) Find the reduced row-echelon form of the augmented matrix.

(e) Find the solution of the linear system.

Solution:

(a)

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 0 \\ 2 & 3 & -1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 2 & 2 & 9 \\ 1 & 1 & 0 & 1 \\ 2 & 3 & -1 & 1 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & 2 & 2 & 9 \\ 1 & 1 & 0 & 1 \\ 2 & 3 & -1 & 1 \end{bmatrix} \underset{\xrightarrow{-2r_1+r_3}}{\xrightarrow{-r_1+r_2}} \begin{bmatrix} 1 & 2 & 2 & 9 \\ 0 & -1 & -2 & -8 \\ 0 & -1 & -5 & -17 \end{bmatrix} \xrightarrow{-r_2+r_3} \begin{bmatrix} 1 & 2 & 2 & 9 \\ 0 & -1 & -2 & -8 \\ 0 & 0 & -3 & -9 \end{bmatrix}$$

$$\underset{\xrightarrow{\frac{1}{3}r_3}}{\xrightarrow{-r_2}} \begin{bmatrix} 1 & 2 & 2 & 9 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

(d)

$$\begin{bmatrix} 1 & 2 & 2 & 9 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{bmatrix} \underset{\xrightarrow{r_2-2r_3}}{\xrightarrow{r_1-2r_3}} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{r_1-2r_2} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

(e) If we write out the equivalent linear system using the reduced row-echelon matrix given in part (d) we have,

$$x = -1, y = 2, z = 3$$

The solution to this linear system is the same as the solution to the original linear system because they are equivlaent. Therefore, (x, y, z) = (-1, 2, 3) is the solution to the original linear system.

Exercises

1. Using elementary operations, write each of the following matrices in reduced row-echelon form,

   (a)

$$\begin{bmatrix} -1 & 3 & 1 \\ 0 & -1 & 2 \end{bmatrix}$$

   (b)

$$\begin{bmatrix} 1 & 1 & 4 & 2 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

2. For each of the matrices in #1 determine the solution to the related linear system.

3. Solve each linear system using matrices and row operations. Reduce your augmented matrix to reduced row-echelon form.

   (a)

$$\begin{aligned} 3x - 2y + z &= 6 \\ x - 3y - 2z &= -26 \\ -x + y + z &= 9 \end{aligned}$$

   (b)

$$\begin{aligned} x - y - 3z &- 3 \\ 2x_2 y + z &= -1 \\ -x - y + z &= -1 \end{aligned}$$

(c)

$$\begin{aligned} 2x + 3y + 6z &= 3 \\ x - y - z &= 0 \\ 4x + 3y - 6z &= 2 \end{aligned}$$

(d)

$$\begin{aligned} 2x + y - z &= -6 \\ x - y + 2z &= 9 \\ -x + y + z &= 9 \end{aligned}$$

4. (a) Determine the value of k for which the following system will have an infinite number of solutions,

$$\begin{aligned} x + y + z &= -1 \\ x - y + z &= 2 \\ 3x - y + 3z &= k \end{aligned}$$

(b) For what value(s) of k will this system have no solutions?

(c) Explain why it is not possible for this system to have a unique solution.

5. The linear system below is called a *homogenous system* because all the values to the right of the equal sign are 0.

$$\begin{aligned} 2x - y + z &= 0 \\ x + y + z &= 0 \\ 5x - y + 3z &= 0 \end{aligned}$$

(a) Explain why every homogeneous sytem of equations has at least one solution.

(b) Write the related augmented matrix in reduced row-echelon form and explain the meaning of this result.

