Matrices

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Matrices (Introduction)

Considering a linear system of equations,

$$\begin{aligned} x + 2y + 2z &= 9 \\ x + y &= 1 \\ 2x + 3y - z &= 1 \end{aligned}$$

A *matrix* is a box of numbers. For a linear system we have a couple of matrices of interest. The *coefficient matrix* of a linear system, in particular for the linear system above, is given by,

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 0 \\ 2 & 3 & -1 \end{bmatrix}$$

contains all the coefficients for the variables x, y and z in the linea system. Notice the first equation's coefficients corresponds to the first row of the matrix, and so on. The *augmented matrix* for the linear system is given by,

$$\begin{bmatrix} 1 & 2 & 2 & 9 \\ 1 & 1 & 0 & 1 \\ 2 & 3 & -1 & 1 \end{bmatrix}$$

Notice that the augmented matrix looks like the coefficient matrix with one extra right most column consisting of the values to the right of the equal signes for the equations in the linear system. Notice that in both matrices the columns correpond to a particular variable in the linear system. That is, the first column of the matrix corresponds to all the coefficient for the variable x and so on. This means that the equations in the linear system shoudl be written with all the variables in the same order.

Recall the elementary operations on a linear system. These elementary operations have equivlaent *elementary row operations on matrices.*

Elementary row operations for matrices

1. **Multipy** a row by a non-zero constant.
2. **Interchange** any pair of rows.
3. **Add** a multiple of one row to a second row to replace the second row.

Note: A row in the matrix represents an equation in the linear system.

Elementary row operations may be performed on a matrix. Any matrix that results from elementary row operations is *row equivalent* to the original matrix. Suppose elementary row operations are performed on a matrix until the following matrix structure occurs,

$$\begin{bmatrix} \# & \# & \# & \# \\ 0 & \# & \# & \# \\ 0 & 0 & \# & \# \end{bmatrix}$$

where # represents any non-zero number. This final structure is called *row echelon form*.

Properties of a matrix in row-echelon form

1. All rows that are all zeros are moved to the bottom of the matrix.
2. The first non-zero entry in a row, called the *leading entry*, must contain zeros below it.

We can take the row-echelon form a reduce it further to the following structure,

$$\begin{bmatrix} 1 & 0 & 0 & \# \\ 0 & 1 & 0 & \# \\ 0 & 0 & 1 & \# \end{bmatrix}$$

This form is called *reduced row-echelon form*.

Properties of a matrix in reduced row-echelon form

1. The matrix is in row-echelon form.
2. The first non=zero entry in each row is a 1. This 1 is called a *leading 1*.
3. Any column containing a leading 1 has all other column entries equal to 0.

Example

Given the following linear system,

$$\begin{aligned} x + 2y + 2z &= 9 \\ x + y &= 1 \\ 2x + 3y - z &= 1 \end{aligned}$$

(a) Find the coefficent matrix for the linear system.

(b) Find the augmented matrix.

(c) Use elementary row operations to find the row-echelon form of the augmented matrix.

(d) Find the reduced row-echelon form of the augmented matrix.

(e) Find the solution of the linear system.

Solution:

(a)

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 0 \\ 2 & 3 & -1 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 2 & 2 & 9 \\ 1 & 1 & 0 & 1 \\ 2 & 3 & -1 & 1 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & 2 & 2 & 9 \\ 1 & 1 & 0 & 1 \\ 2 & 3 & -1 & 1 \end{bmatrix} \underset{\xrightarrow{-2r_1+r_3}}{\xrightarrow{-r_1+r_2}} \begin{bmatrix} 1 & 2 & 2 & 9 \\ 0 & -1 & -2 & -8 \\ 0 & -1 & -5 & -17 \end{bmatrix} \xrightarrow{-r_2+r_3} \begin{bmatrix} 1 & 2 & 2 & 9 \\ 0 & -1 & -2 & -8 \\ 0 & 0 & -3 & -9 \end{bmatrix}$$

$$\underset{\xrightarrow{\frac{1}{3}r_3}}{\xrightarrow{-r_2}} \begin{bmatrix} 1 & 2 & 2 & 9 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

(d)

$$\begin{bmatrix} 1 & 2 & 2 & 9 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{bmatrix} \underset{\xrightarrow{r_2-2r_3}}{\xrightarrow{r_1-2r_3}} \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{r_1-2r_2} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

(e) If we write out the equivalent linear system using the reduced row-echelon matrix given in part (d) we have,

$$x = -1, y = 2, z = 3$$

The solution to this linear system is the same as the solution to the original linear system because they are equivlaent. Therefore, (x, y, z) = (-1, 2, 3) is the solution to the original linear system.

Exercises

1. Write an augmented matrix for each system of equations.

   (a)

$$\begin{aligned} x + 2y - z &= -1 \\ -x + 3y - 2z &= -1 \\ 3y - 2z &= -3 \end{aligned}$$

   (b)

$$\begin{aligned} 2x - z &= 1 \\ 2y - z &= 16 \\ -3x + y &= 10 \end{aligned}$$

   (c)

$$\begin{aligned} 2x - y - z &= -2 \\ x - y + 4z &= -1 \\ -x - y &= 13 \end{aligned}$$

2. Reduce the following augmented matrix to row-echelon form. Note: Ensure there ar eno fractions in the final matrix.

$$\begin{bmatrix} 2 & 1 & 6 & 0 \\ 0 & -2 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{bmatrix}$$

3. (a) Write the following aumgneted matrix in row-echelon form. Ensure that every number in the matrix is an integer.

$$\begin{bmatrix} -1 & 0 & 1 & 2 \\ 0 & -1 & 2 & 0 \\ \frac{1}{2} & -\frac{3}{4} & -2 & \frac{1}{3} \end{bmatrix}$$

(b) Solve the system of equations corresponding to the matrix derived in (a).

4. Write the system of equations that corresponds to each augmente matrix.

(a)

$$\begin{bmatrix} 1 & -2 & -1 \\ 2 & -3 & 1 \\ 2 & -1 & 0 \end{bmatrix}$$

(b)

$$\begin{bmatrix} -2 & 0 & -1 & 0 \\ 1 & -2 & 0 & -4 \\ 0 & 1 & 2 & -3 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -2 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

5. The following matrices represent the row-echelon form of a matrix. Write out the solution, if it exists, of the equivalent linear system.

(a)

$$\begin{bmatrix} -2 & 1 & 6 \\ 0 & -5 & 15 \end{bmatrix}$$

(b)

$$\begin{bmatrix} -1 & 3 & 1 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 13 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & -1 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

6. For each of the matrices below,

   i) Determine whether the matrix is in row-echelon form. If not, use elementary row operations to reduce the matrix to row-echelon form.

   ii) Write the solution to the linear system corresponding to the augmented matrix given, if a solution exists.

   iii) Give a geometric interpretation of the solution, if it exists.

   (a)

$$\begin{bmatrix} -1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

   (b)

$$\begin{bmatrix} -1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -6 \end{bmatrix}$$

   (c)

$$\begin{bmatrix} 1 & -4 & 1 & 0 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

7. Solve the linear system using matrices and interpret the result geometrically.

(a)

$$\begin{aligned} x + y + z &= 0 \\ 2x + 3y + z &= 0 \\ -3x - 2y - 4z &= 0 \end{aligned}$$

(b)

$$\begin{aligned} x + 3y + 4z &= 4 \\ -x + 3y + 8z &= -4 \\ x - 3y - 4z &= -4 \end{aligned}$$

(c)

$$\begin{aligned} x - y &= -500 \\ 2y - z &= 3500 \\ x - z &= 2000 \end{aligned}$$

8. A system of equations has the following augmented matrix,

$$\begin{bmatrix} a & 1 & 1 & a \\ 1 & a & 1 & a \\ 1 & 1 & a & a \end{bmatrix}$$

Determine the values of a if the corresponding system of equation has,

(a) no solutions

(b) an infinite number of solutions

(c) exactly one solution