Logarithmic Differentiation

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Logarthimic Differentiation

Let's consider an example.

Example

For $y = x^x,\ x > 0$. Determine $\frac{dy}{dx}$.

Solution:

$$\begin{aligned}
y &= x^x, \quad \text{Take the natural logarithm, ln, of both sides.} \\
\ln y &= \ln x^x \\
\ln y &= x \ln x, \quad \text{Differentiate w.r.t. } x \text{ on both sides.} \\
\frac{1}{y}\frac{dy}{dx} &= \ln x + \frac{x}{x} \\
\frac{1}{y}\frac{dy}{dx} &= \ln x + 1 \\
\frac{dy}{dx} &= y(\ln x + 1) \\
\therefore \frac{dy}{dx} &= x^x(\ln x + 1)
\end{aligned}$$

Example

Differentiate $y = x^n$.

Solution:

$$\begin{aligned}
y &= x^n \\
\ln y &= \ln x^n, \quad \text{Take the natural logarithm, ln, of both sides.} \\
\ln y &= n \ln x \\
\frac{1}{y}\frac{dy}{dx} &= \frac{n}{x} \\
\frac{dy}{dx} &= \frac{ny}{x} \\
&= \frac{nx^n}{x} \\
\therefore \frac{dy}{dx} &= nx^{n-1}
\end{aligned}$$

Example

Differentiate $y = (x^2 + 3)^x$.

Solution:

$$\begin{aligned}
y &= (x^2 + 3)^x \\
\ln y &= \ln(x^2 + 3)^x \\
\ln y &= x \ln(x^2 + 3) \\
\frac{1}{y}\frac{dy}{dx} &= \ln(x^2 + 3) = \frac{x(2x)}{x^2 + 3} \\
&= \ln(x^2 + 3) + \frac{2x^2}{x^2 + 3} \\
\frac{dy}{dx} &= y\left(\ln(x^2 + 3) + \frac{2x^2}{x^2 + 3}\right) \\
\frac{dy}{dx} &= (x^2 + 3)^x\left(\ln(x^2 + 3) + \frac{2x^2}{x^2 + 3}\right)
\end{aligned}$$

Example

Differentiate

$$y = \frac{(x^4+1)\sqrt{x+2}}{(2x^2+2x+1)}$$

Solution: We'll use logarithmic differentiation.

$$\begin{aligned}
y &= \frac{(x^4+1)\sqrt{x+2}}{(2x^2+2x+1)} \\
\ln y &= \ln\left(\frac{(x^4+1)\sqrt{x+2}}{(2x^2+2x+1)}\right) \\
\ln y &= \ln *x^4+1) + \ln\sqrt{x+2} - \ln(2x^2+2x+1) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{4x^3}{x^4+1} + \frac{1}{2}\frac{1}{x+2} - \frac{4x+2}{2x^2+2x+1} \\
\frac{dy}{dx} &= y\left(\frac{4x^3}{x^4+1} + \frac{1}{2}\frac{1}{x+2} - \frac{4x+2}{2x^2+2x+1}\right) \\
&= \frac{(x^4+1)\sqrt{x+2}}{(2x^2+2x+1)}\left(\frac{4x^3}{x^4+1} + \frac{1}{2}\frac{1}{x+2} - \frac{4x+2}{2x^2+2x+1}\right)
\end{aligned}$$

Exercises

1. Find the derivative of the following. Recall

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

   a) $y = \ln|x|$

   b) $y = \ln|2x + 1|$

   c) $y = x^2 \ln|x|$

2. Find the slope of the tangent,

   a) To the curve $xe^y + y \ln x = 2$ at $(1, \ln 2)$.

   b) to the curve $\ln\sqrt{xy} = 0$ at the point $(1/3, 3)$.