Logarithmic Differentiation

RaiseMyMarks.com

2021

Logarthimic Differentiation

Let's consider an example.

Example

For $y = x^x,\ x > 0$. Determine $\frac{dy}{dx}$.

Solution:

$$\begin{aligned}
y &= x^x, \quad \text{Take the natural logarithm, ln, of both sides.} \\
\ln y &= \ln x^x \\
\ln y &= x \ln x, \quad \text{Differentiate w.r.t. } x \text{ on both sides.} \\
\frac{1}{y}\frac{dy}{dx} &= \ln x + \frac{x}{x} \\
\frac{1}{y}\frac{dy}{dx} &= \ln x + 1 \\
\frac{dy}{dx} &= y(\ln x + 1) \\
\therefore \frac{dy}{dx} &= x^x(\ln x + 1)
\end{aligned}$$

Example

Differentiate $y = x^n$.

Solution:

$$\begin{aligned}
y &= x^n \\
\ln y &= \ln x^n, \quad \text{Take the natural logarithm, ln, of both sides.} \\
\ln y &= n \ln x \\
\frac{1}{y}\frac{dy}{dx} &= \frac{n}{x} \\
\frac{dy}{dx} &= \frac{ny}{x} \\
&= \frac{nx^n}{x} \\
\therefore \frac{dy}{dx} &= nx^{n-1}
\end{aligned}$$

Example

Differentiate $y = (x^2 + 3)^x$.

Solution:

$$\begin{aligned}
y &= (x^2 + 3)^x \\
\ln y &= \ln(x^2 + 3)^x \\
\ln y &= x \ln(x^2 + 3) \\
\frac{1}{y}\frac{dy}{dx} &= \ln(x^2 + 3) = \frac{x(2x)}{x^2 + 3} \\
&= \ln(x^2 + 3) + \frac{2x^2}{x^2 + 3} \\
\frac{dy}{dx} &= y\left(\ln(x^2 + 3) + \frac{2x^2}{x^2 + 3}\right) \\
\frac{dy}{dx} &= (x^2 + 3)^x\left(\ln(x^2 + 3) + \frac{2x^2}{x^2 + 3}\right)
\end{aligned}$$

Example

Differentiate

$$y = \frac{(x^4+1)\sqrt{x+2}}{(2x^2+2x+1)}$$

Solution: We'll use logarithmic differentiation.

$$\begin{aligned}
y &= \frac{(x^4+1)\sqrt{x+2}}{(2x^2+2x+1)} \\
\ln y &= \ln\left(\frac{(x^4+1)\sqrt{x+2}}{(2x^2+2x+1)}\right) \\
\ln y &= \ln *x^4+1) + \ln\sqrt{x+2} - \ln(2x^2+2x+1) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{4x^3}{x^4+1} + \frac{1}{2}\frac{1}{x+2} - \frac{4x+2}{2x^2+2x+1} \\
\frac{dy}{dx} &= y\left(\frac{4x^3}{x^4+1} + \frac{1}{2}\frac{1}{x+2} - \frac{4x+2}{2x^2+2x+1}\right) \\
&= \frac{(x^4+1)\sqrt{x+2}}{(2x^2+2x+1)}\left(\frac{4x^3}{x^4+1} + \frac{1}{2}\frac{1}{x+2} - \frac{4x+2}{2x^2+2x+1}\right)
\end{aligned}$$

Exercises

Differentiate.

a) $y = \ln(5x + 8)$

b) $s = \ln(t^3 - 2t^2 + 5)$

c) $w = \ln\sqrt{z^2 + 3z}$

d) $y = \ln\sqrt{x + 1}$

e) $f(x) = x \ln x$

f) $y = \frac{\ln x}{x^2}$

g) $v = e^t \ln t$

h) $y = (\ln x)^3$

i) $x = \frac{e^t}{\ln t}$

j) $f(x) = \ln\left(\frac{x^2+1}{x-1}\right)$

k) $g(z) = \ln(e^{-z} + ze^{-z})$

n) $f(x) = \ln(x^2 + 1)$

l) $h(u) = e^{\sqrt{u}} \ln \sqrt{u}$

o) $f(x) = (x^2 + 1)^{-1} \ln(x^2 + 1)$

m) $g(x) = e^{2x=1} \ln(2x - 1)$