

Definition of a Limit

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2021

Definition of a Limit

In words the *limit* of a function is a value that the function approaches as the independent variable, usually denoted by x , tends to a particular value. This particular value is either a finite value a say, or infinity, ∞ . Below is the definition of a *limit* in mathematical notation,

$$\lim_{x \rightarrow a} f(x) = L,$$

where L exists, means,

$$\forall \epsilon > 0, \exists \delta > 0 \ni, |x - a| < \delta \implies |f(x) - L| < \epsilon \quad (1)$$

The notation $x \rightarrow a$ means that x is approaching a from any direction. However, in some cases it may be the case that the limit only exists when a is being approached from either the left or right. In these cases the limit is defined below,

Right hand Limit Definition

$$\lim_{x \rightarrow a^+} f(x) = L$$

is defined as,

$$\forall \epsilon > 0, \exists \delta > 0 \ni, x - a > \delta \implies |f(x) - L| < \epsilon \quad (2)$$

In this above limit, x is approaching a from the positive side of a or from the right hand side. Now let's consider the definition of a limit when x is approaching a from the left hand side.

Left hand limit Definition

$$\lim_{x \rightarrow a^-} f(x) = L$$

is defined as,

$$\forall \epsilon > 0, \exists \delta > 0 \ni, x - a < \delta \implies |f(x) - L| < \epsilon \quad (3)$$

So far we've considered limits as x approaches a finite value a . In some cases the limit occurs when x approaches infinity. Let's take a look at the definition of the limit as x tends to positive infinity and then negative infinity.

Limit as x tends to infinity (Horizontal asymptotes)

$$\lim_{x \rightarrow +\infty} f(x) = L$$

is defined by,

$$\forall \epsilon > 0 \exists N > 0 \ni, x > N \implies |f(x) - L| < \epsilon \quad (4)$$

and,

$$\lim_{x \rightarrow -\infty} f(x) = L$$

is defined by,

$$\forall \epsilon > 0 \exists N < 0 \ni, x < N \implies |f(x) - L| < \epsilon \quad (5)$$

So far we have considered finite limits L . Let's now consider infinite limits, that is when our L is replaced by negative or positive infinity.

Infinite Limits

$$\lim_{x \rightarrow a} f(x) = +\infty$$

is defined as

$$\forall N > 0, \exists \delta > 0 \ni, |x - a| < \delta \implies f(x) > N \quad (6)$$

or

$$\lim_{x \rightarrow a} f(x) = -\infty$$

is defined a

$$\forall N < 0, \exists \delta > 0 \ni, |x - a| < \delta \Rightarrow f(x) < N \quad (7)$$

Above we considered the situation where x approaches a from any direction, left or right. However, we can also have x approaching a from just the left or just the right. These definitions are below.

$$\lim_{x \rightarrow a^+} f(x) = +\infty$$

is defined as,

$$\forall N > 0, \exists \delta > 0 \ni, x - a > \delta \Rightarrow f(x) > N \quad (8)$$

and

$$\lim_{x \rightarrow a^-} f(x) = +\infty$$

is defined as,

$$\forall N > 0, \exists \delta > 0 \ni, x - a < \delta \Rightarrow f(x) > N \quad (9)$$

And for the negative limit,

$$\lim_{x \rightarrow a^+} f(x) = -\infty$$

is defined as,

$$\forall N < 0, \exists \delta > 0 \ni, x - a > \delta \Rightarrow f(x) < N \quad (10)$$

and

$$\lim_{x \rightarrow a^-} f(x) = -\infty$$

is defined as,

$$\forall N < 0, \exists \delta > 0 \ni, x - a < \delta \Rightarrow f(x) < N \quad (11)$$

Examples

Find the following limit.

$$\lim_{x \rightarrow 0} \frac{x^4 - 5x}{x^3 + 2x}$$

Solution:

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{x^4 - 5x}{x^3 + 2x} \\ &= \lim_{x \rightarrow 0} \frac{x(x^3 - 5)}{x(x^2 + 2)} \\ &= \lim_{x \rightarrow 0} \frac{x^3 - 5}{x^2 + 2} \\ &= \frac{0 - 5}{0 + 2} \\ &= -\frac{5}{2} \end{aligned}$$

Therefore,

$$\lim_{x \rightarrow 0} \frac{x^4 - 5x}{x^3 + 2x} = -\frac{5}{2}.$$

Examples

Find the limit below,

$$\lim_{x \rightarrow \infty} \frac{3x^2 - 2x + 1}{5x^3 + 4}$$

Solution:

$$\begin{aligned} & \lim_{x \rightarrow \infty} \frac{3x^2 - 2x + 1}{5x^3 + 4} \\ &= \lim_{x \rightarrow \infty} \frac{x^3 \left(3 - 2\frac{1}{x^2} + \frac{1}{x^3}\right)}{x^3 \left(5 + 4\frac{1}{x^3}\right)} \\ &= \lim_{x \rightarrow \infty} \frac{3 - \frac{2}{x^2} - \frac{1}{x^2}}{5 + \frac{4}{x^3}} \\ &= \lim_{x \rightarrow \infty} \frac{3 - 0 - 0}{5 + 0} \\ &= \frac{3}{5} \end{aligned}$$

Therefore,

$$\lim_{x \rightarrow \infty} \frac{3x^2 - 2x + 1}{5x^3 + 4} = \frac{3}{5}$$

Exercises

Find the following limits,

1.

$$\lim_{x \rightarrow 3} \frac{1}{(x - 3)^2}$$

5.

$$\lim_{x \rightarrow \infty} \frac{3 - x}{x^2 + 2x}$$

2.

$$\lim_{x \rightarrow \infty} \frac{1}{x + 1}$$

6.

$$\lim_{x \rightarrow 1^+} \frac{1}{x - 1}$$

3.

$$\lim_{x \rightarrow \infty} \frac{2x^3 - 5}{7x^3 + x - 1}$$

7.

$$\lim_{x \rightarrow 2^+} \frac{1}{x + 2}$$

4.

$$\lim_{x \rightarrow \infty} \frac{x^3}{x^2 - 4x}$$

8.

$$\lim_{x \rightarrow \infty} \frac{5x^2}{x - 3x^2}$$