Derivatives

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2021

Derivative

The derivative of a function f at a point $x = s$ may be though tof as the slope of the tangent to the cuve at the point $x = a$. Or the rate of change of the function f at the point $x = a$. How is the derivative defined in mathematical notation?

Definition of the derivative

The derivative of a function f at the point $x = a$ is given by,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided that the limit exits. Or,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

or the derivative of f w.r.t. x is given by,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists.

Examples

Let's consider some examples. Find the derivative of $f(x) = x^2$ at $x = -3$.

Solution:

$$
\begin{aligned}
f'(-3) &= \lim_{h\to 0} \frac{f(-3+h)-f(-3)}{h} \\
&= \lim_{h\to 0} \frac{(-3+h)^2-(-3)^2}{h} \\
&= \lim_{h\to 0} \frac{9-6h+h^2-9}{h} \\
&= \lim_{h\to 0} \frac{h(h-6)}{h} \\
&= -6
\end{aligned}
$$

Example

Let's find $f'(-3)$ using the second definition of the derivative.

Solution:

$$
\begin{aligned}
f'(-3) &= \lim_{x\to -3} \frac{f(x)-f(-3)}{x+3} \\
&= \lim_{x\to -3} \frac{x^2-(-3)^2}{x+3} \\
&= \lim_{x\to -3} \frac{x^2-9}{x+3} \\
&= \lim_{x\to -3} \frac{(x-3)(x+3)}{(x+3)} \\
&= -6
\end{aligned}
$$

Example

Let's find the derivative of $f(x) = x^2$ at an arbitrary value for x.

Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2-x^2}{h} \\ &= \lim_{h\to 0} \frac{(x+h-x)(x+h+x)}{h} \\ &= \lim_{h\to 0} \frac{h(2x+h)}{h} \\ &= 2x \end{aligned}$$

Example

Let's consider the following example. Find $f'(t)$ for the function $f(t) = \sqrt{t}, t \geq 0$.

Solution:

$$\begin{aligned} f'(t) &= \lim_{h\to 0} \frac{f(t+h)-f(t)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{t+h}-\sqrt{t}}{h} \\ &= \lim_{h\to 0} \left(\frac{\sqrt{t+h}-\sqrt{t}}{h}\right)\left(\frac{\sqrt{t+h}+\sqrt{t}}{\sqrt{t+h}+\sqrt{t}}\right) \\ &= \lim_{h\to 0} \frac{t+h-t}{h(\sqrt{t+h}+\sqrt{t})} \\ &= \lim_{h\to 0} \frac{h}{h(\sqrt{t+h}-\sqrt{t}} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{t+h}+\sqrt{t}} \\ &= \frac{1}{2\sqrt{t}},\ t > 0 \end{aligned}$$

Exercises

Use the definition of the derivative to determine the derivative.

a) $f(x) = x^2 + 3x$

b) $f(x) = \frac{3}{x+2}$

c) $f(x) = \sqrt{3x+2}$

d) $f(x) = \frac{1}{x^2}$

e) $y = c$

f) $y = x$

g) $y = mx + b$, where m and b are constants

h) $y = ax^2 + bx + c$, where a, b and c are constants.