Optimization Problems

RaiseMyMarks.com

2021

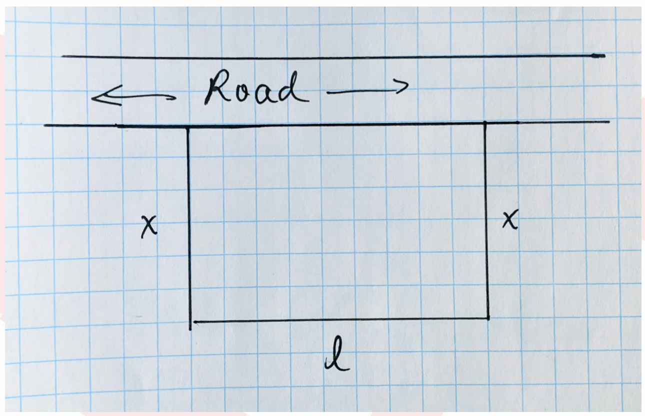
Optimization Problems

What is an optimization problem? What does optimization mean? The procedure of determining the maximum or minimum of a quantity is called *optimization* of that quantity. Let's consider an example to better understand optimization.

Example

A farmer has $800m$ of fencing and wants to enclose a rectangular filed. One side is closed off by a fence by a road. So only 3 sides of the filed need to be fenced in. What are the dimensions of the filed with greatest area?

Solutions: Let's draw the area to be fenced in with the information provided.



We have $800m$ of fencing so

$$800 = 2x + l \text{ or } 800 - 2x = l$$

The area of a rectangle is given by

$$A = xl = x(800 - 2x)$$

or

$$A(x) = 800x - 2x^2.$$

We know how to find the maximum of a function. Start by finding the point where the maximum or minimun may occur. We need to find the *critical points* of our function $A(x) = 800x - 2x^2$. We do this by solving $A'(x) = 0$.

$$\begin{aligned} A'(x) &= 800 - 4x = 0 \\ \frac{800}{4} &= \frac{4x}{4} \\ 200 &= x \end{aligned}$$

The width of the field with the maximum area enclosed by $800m$ is $200m = x$. Now we need to find the length. We use the original relationship for the fencing, $800 = 2x + l$.

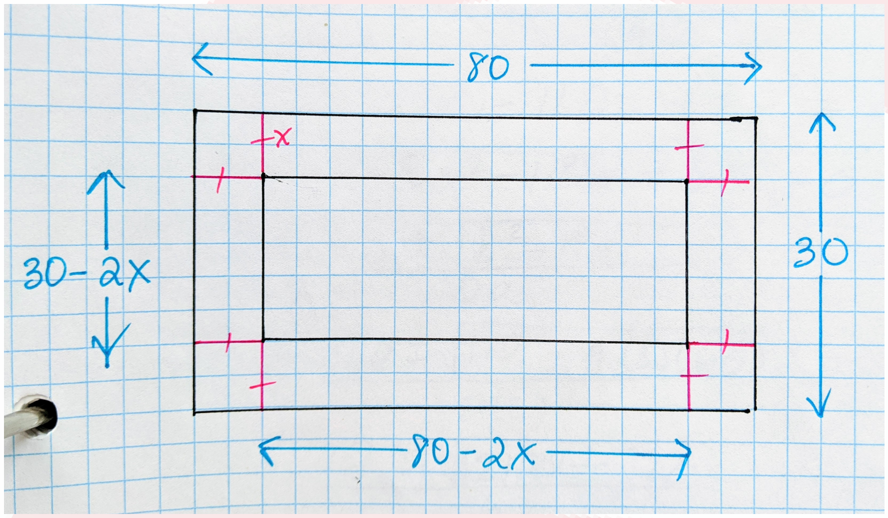
$$\begin{aligned} l &= 800 - 2x = 800 - 2(200) \\ &= 800 - 400 = 400m \end{aligned}$$

Therefore, the dimensions of the fenced area are $200m \times 400m$. Let's try another example.

Example

A piece of sheet metal, $80cm \times 30cm$ is to be used to make a rectangule box with an open top. Find the dimensions that will give the box the largest volume.

Solutions:



$$\begin{aligned} \text{volumne} &= lwh \\ &= (80-2x)(30-2x)x \\ &= (240-160x-60x-60x+4x^2)x \\ &= (240-220x+4x^2)x \\ V(x) &= 240x-220x^2+4x^3 \end{aligned}$$

We need to find the critical points of the function for the volume $V(x)$.

$$\begin{aligned} V'(x) &= \frac{240-440x+12x^2}{4} = \frac{0}{4} \\ 0 &= 60-110x+3x^2 \\ 0 &= 3x^2-110x+60 \end{aligned}$$

We need to use the quadratic formula to find the zeros of the quadratic. Recall the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

From here we have,

$$\begin{aligned} x &= \frac{110 \pm \sqrt{110^2 - 4(3)(60)}}{2(3)} \\ &= \frac{110 \pm \sqrt{12100 - 720}}{6} \\ &= \frac{110 \pm \sqrt{11380}}{6} \\ &= \frac{110 \pm 110668}{6} \\ \therefore x &= 36.11 \text{ or } 0.55 \end{aligned}$$

From here we see that $x = 36.11cm$ is not a reasonable height so$x = 0, 0.55cm$ are the possible heights. Therefore,

$$\begin{aligned} v''(x) &= -440 + 24(0.55) \\ &= -440 + 13.2 \\ &= -426.8 \end{aligned}$$

Therefore, the maximum volume occurs when $x = 0.55cm$ since $V'(0.55) < 0$. The maximum volume of the box is,

$$v(0.55) = 240(0.55) - 220(0.55)^2 + 4(0.55)^3 = 66.12cm^3$$

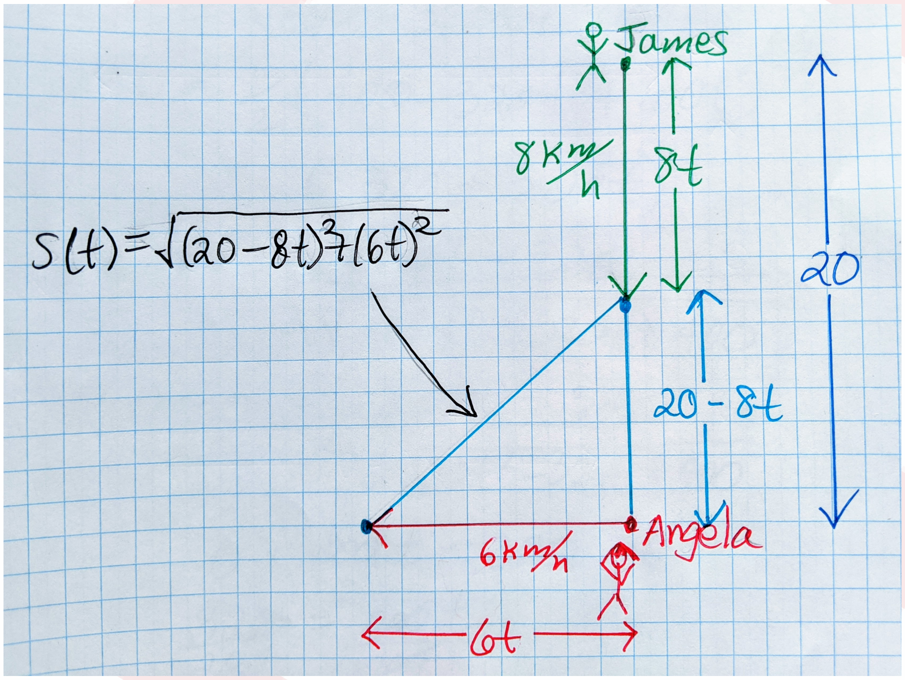
with dimensions,

$$80 - 2(0.55) = 78.9cm, \quad 30 - 2(0.55) = 78.9cm \text{ and } 0.55cm$$

Example

James and Angela both run each morning. James' house is 20km north of Angela's. Both start at 9am. James leaves his house and runs south at 8km/h. At the same time Angela leaves her house and runs west at 6km/h. When are James and Angela closest after 2.5h?

Solutions:



$$
\begin{aligned}
s(t) &= \sqrt{(20-8t)^2+(6t)^2} \\
&= \sqrt{400-320t+64t^2+36t^2} \\
&= \sqrt{400-320t+100t^2} \\
s'(t) &= \frac{1}{2}(400-320t+100t^2)^{-1/2}(-320+200t) \\
&= \frac{200t-320}{2\sqrt{100t^2-320t+400}} \\
s''(t) &= 0 = \frac{200t-320}{2\sqrt{100t^2-320t+400}} \\
0 &= 200t-320 \\
\frac{320}{200} &= \frac{200t}{200} \\
1.6 &= t
\end{aligned}
$$

The domain for t is $0 \le t \le 2.5$. We need to check which value of t, $t = 0, 1.6$ or 2.5 gives the smalles distance $s(t)$ value.

$$\begin{aligned} s(0) &= \sqrt{(20-8(0))^2 + (6(0))^2} \\ &= \sqrt{(20)^2 + 0} \\ &= 20 \\ s(1.6) &= \sqrt{(20-8(1.6))^2 + (6(1.6))^2} \\ &= \sqrt{51.84 + 92.16} \\ &= \sqrt{144} \\ &= 12 \\ s(2.5) &= \sqrt{(20-8(2.5))^2 + (6(2.5))^2} \\ &= \sqrt{0 + 15^2} \\ &= 15 \end{aligned}$$

At time $t = 1.6$ hours, James and Angela are at their closest after 1.6 hours of their 2.5 hour run. So at time 10:36am Angela and James are at their closest.

Exercises

1. Find two positive real numbers whose sume is 105 and whose product is a maximum.