L'Hôpital's Rule
Indeterminate Forms of a Limit

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2021

Indeterminate forms of Limits

It is possible when evaluating a limit we run into some problems. For example, when we go through the steps of evaluating the limit and we end up in one of the following situations,

$$\frac{\infty}{\infty},\ \frac{0}{0},\ (0)(\pm\infty),\ 1^{\infty},\ 0^{0},\ \infty^{0},\ \infty - \infty \tag{1}$$

Any of these forms in (1) is referred to as an *indeterminate form* of a limit. When faced with any of these indeterminate forms the following "rule" is performed to hopefully eliminate the indeterminate form. This rule is called **l'Hôpital's Rule** given below,

l'Hôpital's Rule

Suppose,

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{0}{0} \text{ or } \lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$$

where a is any real number ∞ or $+\infty$. Then,

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example

Find the following limit

$$\lim_{x\to 0}\frac{\sin x}{x}$$

Solution:

$$\begin{aligned} & \lim_{x\to 0}\frac{\sin x}{x} \\ = \; & \frac{0}{0}, \text{ indeterminatne form so apply l'Hôpital's rule} \\ = \; & \lim_{x\to 0}\frac{\cos x}{1} \\ = \; & 1 \end{aligned}$$

Example

Evaluate the following limit,

$$\lim_{x\to 0^+} x\ln x$$

Solution:

$$\begin{aligned} & \lim_{x\to 0^+} x\ln x \\ = \; & 0\cdot\infty \end{aligned}$$

While $0\cdot\infty$ is an indeterminate form it is not in one of the forms that is found in l'Hôpital's rule. So we need to try and rearrange the function so that we obtain either $\frac{0}{0}$ or $\frac{\infty}{\infty}$ as the indeterminate form.

$$
\begin{aligned}
& \lim_{x\to 0^+} x\ln x \\
= & \lim_{x\to 0^+} \frac{\ln x}{1/x} \text{ now we will evaluate the limit of this form of the function ad see what} \\
= & \lim_{x\to 0^+} \frac{1/x}{-1/x^2} \\
= & \frac{\infty}{\infty} \text{ Now we can apply l'Hôpital's Rule} \\
= & \lim_{x\to 0^+} \frac{1}{x}\left(-\frac{x^2}{1}\right) \\
= & \lim_{x\to 0^+} (-x) \\
= & 0
\end{aligned}
$$

Note:

$$f(x)g(x) = \frac{g(x)}{1/f(x)} = \frac{f(x)}{1/g(x)}$$

Example

Evaluate the following limit,

$$\lim_{x\to a} \frac{x-a}{x^2-a^2}$$

Solution:

$$
\begin{aligned}
&= \lim_{x\to a} \frac{x-a}{x^2-a^2} \\
&= \lim_{x\to a} \frac{x-a}{(x-a)(x+a)} \\
&= \lim_{x\to a} \frac{1}{x+a} \\
&= \frac{1}{2a}
\end{aligned}
$$

Exercises

Find the following limits,

1. $$\lim_{x\to\pi/2}\frac{\cos x}{\pi/2-x}$$

2. $$\lim_{x\to 0} frace^{x}-x-1x^{2}$$

3. $$\lim_{x\to 0}\frac{\tan x}{\sqrt{x}}$$

4. $$\lim_{x\to 0}(x+1)^{1/x}$$

5. $$\lim_{x\to 0}\frac{\sin x - x}{x^2}$$

6. $$\lim_{x\to 0^+} x\ln(x^4)$$

7. $$\lim_{x\to 1^+}\left(\frac{1}{\ln x}-\frac{1}{1-x}\right)$$

8.
$$\lim_{x\to\infty}\left(\frac{x+7}{x+3}\right)^x$$

9.
$$\lim_{x\to 0}\frac{3^x-2^x}{x}$$