

L'Hôpital's Rule
Indeterminate Forms of a Limit

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2021

Indeterminate forms of Limits

It is possible when evaluating a limit we run into some problems. For example, when we go through the steps of evaluating the limit and we end up in one of the following situations,

$$\frac{\infty}{\infty}, \frac{0}{0}, (0)(\pm\infty), 1^\infty, 0^0, \infty^0, \infty - \infty \quad (1)$$

Any of these forms in (1) is referred to as an *indeterminate form* of a limit. When faced with any of these indeterminate forms the following “rule” is performed to hopefully eliminate the indeterminate form. This rule is called **l’Hôpital’s Rule** given below,

l’Hôpital’s Rule

Suppose,

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{0}{0} \text{ or } \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}$$

where a is any real number ∞ or $+\infty$. Then,

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

Example

Find the following limit

$$\lim_{x \rightarrow 0} \frac{\sin x}{x}$$

Solution:

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{\sin x}{x} \\ &= \frac{0}{0}, \text{ indeterminate form so apply l'Hôpital's rule} \\ &= \lim_{x \rightarrow 0} \frac{\cos x}{1} \\ &= 1 \end{aligned}$$

Example

Evaluate the following limit,

$$\lim_{x \rightarrow 0^+} x \ln x$$

Solution:

$$\begin{aligned} & \lim_{x \rightarrow 0^+} x \ln x \\ &= 0 \cdot \infty \end{aligned}$$

While $0 \cdot \infty$ is an indeterminate form it is not in one of the forms that is found in l'Hôpital's rule. So we need to try and rearrange the function

so that we obtain either $\frac{0}{0}$ or $\frac{\infty}{\infty}$ as the indeterminate form.

$$\begin{aligned} & \lim_{x \rightarrow 0^+} x \ln x \\ = & \lim_{x \rightarrow 0^+} \frac{\ln x}{1/x} \text{ now we will evaluate the limit of this form of the function and see what} \\ = & \lim_{x \rightarrow 0^+} \frac{1/x}{-1/x^2} \\ = & \frac{\infty}{\infty} \text{ Now we can apply l'Hôpital's Rule} \\ = & \lim_{x \rightarrow 0^+} \frac{1}{x} \left(-\frac{x^2}{1} \right) \\ = & \lim_{x \rightarrow 0^+} (-x) \\ = & 0 \end{aligned}$$

Note:

$$f(x)g(x) = \frac{g(x)}{1/f(x)} = \frac{f(x)}{1/g(x)}$$

Example

Evaluate the following limit,

$$\lim_{x \rightarrow a} \frac{x - a}{x^2 - a^2}$$

Solution:

$$\begin{aligned} &= \lim_{x \rightarrow a} \frac{x - a}{x^2 - a^2} \\ &= \lim_{x \rightarrow a} \frac{x - a}{(x - a)(x + a)} \\ &= \lim_{x \rightarrow a} \frac{1}{x + a} \\ &= \frac{1}{2a} \end{aligned}$$

Exercises

Find the following limits,

1.

$$\lim_{x \rightarrow 0} \frac{\sin x}{x}$$

5.

$$\lim_{x \rightarrow \infty} x^{1/x^2}$$

2.

$$\lim_{x \rightarrow \infty} \frac{\ln x}{x^k}$$

6.

$$\lim_{x \rightarrow 0} x^{2/x}$$

3.

$$\lim_{x \rightarrow a} \frac{x - a}{x^2 - a^2}$$

7.

$$\lim_{x \rightarrow 0} \frac{x^2}{1/x}$$

4.

$$\lim_{x \rightarrow 0^+} x^2 \ln x$$

8.

$$\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$$