Annuities

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Annuities

What is an annuity? An annuity is a series of equal payments or deposits earning compound interest. These payments are made at reguar intervals over a fixed period of time. For example, say ou decide to buy a car but you don't have the total purchase price up front. In this case what will most likely be the scenario to make the purchase is you will be given a loan that you will pay off in regular payments, let's say monthly payments, for a certain fixed period time, for example 7 years. These payments will be made at a particular interest rate. This is a common way to pruchase larger priced items. A house is another common larger priced item that is purchased in this wa.

Simple Ordinary Annuity

A **simple annuity** is when the payment period is the same as the compounding period. An *ordinary annuity* is when tthe payments are made at the *end* of the payment period. Let's consider an example.

Example

Fozzy is saving for the down payment on a new car. Fozzy plans to deposit $100 at the end of each month into an account that pays 5% interest per year, compounded monthly. Determine the amount Fozzy will have afer 2 years.

Solution: We are going to assume this a *simple ordinary annuity* so our payment period equals our pompounding period which is monthly and the payments are at the end of the payment period. Let's have a look at the following table of payments and savings.

Date	Amount
Today	
Month 1	\$100
Month 2	$\$100 + 100\left(1 + \frac{0.05}{12}\right)$
Month 3	$\$100 + 100\left(1 + \frac{0.05}{12}\right) + 100\left(1 + \frac{0.05}{12}\right)^2$
$\vdots$	$\vdots$
Month 24	$\$100 + 100\left(1 + \frac{0.05}{12}\right) + \ldots + 100\left(1 + \frac{0.05}{12}\right)^{23}$

Notice that,

$$100 + 100\left(1 + \frac{0.05}{12}\right) + \ldots + 100\left(1 + \frac{0.05}{12}\right)^{23}$$

is a partial sum of the geometric series

$$100 + 100\left(1 + \frac{0.05}{12}\right) + \ldots + 100\left(1 + \frac{0.05}{12}\right)^{23} + \cdots$$

where

$$a = 100,\ r = 1 + \frac{0.05}{12},\ n = 24.$$

We know that the partial sum or S_n of a geometric series is given by,

$$S_n = a\frac{r^n - 1}{r - 1}$$

Let's look our example and see how we can fit it into the geometric series situation. Our annual interest rate is 5%, the number of payments we will make of the 2 year period is $n = 2 \times 12 = 24$, the monthly compounding interest is $i = 0.05/12$, our monthly payment or annuity is $pmt = \$100$. We want to determine how much money Fozzy will have after 2 years, or the future value, FV.

$$\begin{aligned} FV &= pmt\left(\frac{(1+i)^{24}-1}{(1+i)-1}\right) \\ &= 100\left(\frac{\left(1+\frac{0.05}{12}\right)^{24}-1}{0.05/12}\right) \\ &= \$2518.59 \end{aligned}$$

Therefore, after 2 years Fozzy will have \$2518.59 towards a down paymen on a car.

Let's consider another example.

Example

Scotter deposits \$500 into a savings account on June 1 every year for 5 years. The investment earns 8%/year compounded annually.

a) How much will be in the account after the final deposit?

b) How much interest has Scotter earned?

Solution:

a) We will assume this is a simple ordinary annuity. So, deposits are made at the end of the cpomounding period. Below is a time line of payments and interest earned.

Payment date	Amount in account
June 1, 2020	500
June 1, 2021	$500 + 500(1.08)$
June 1, 2022	$500 + 500(1.08) + 500(1.08)^2$
June 1, 2023	$500 + 500(1.08) + 500(1.08)^2 + 500(1.08)^3$
June 1, 2024	$500 + 500(1.08) + 500(1.08)^2 + 500(1.08)^3 + 500(1.08)^4$

If we rewrite this finally amount, what do we get?

$$500 + 500(1.08) + 500(1.08)^2 + 500(1.08)^3 + 500(1.08)^4$$

We have a geometric series with, $a = 500$, $r = 1.08$ and $n = 5$. The partial sum of a geometric series is given by,

$$S_n = a\left(\frac{r^n - 1}{r - 1}\right)$$

In our situation, $n = 5$, $r = 1 + i$, $i = 0.08$, $a = pmt = \$500$ and $S_5 = FV =$ future value.

$$\begin{aligned} FV &= pmt\left(\frac{(1+i)^5 - 1}{(1+i) - 1}\right) \\ &= 500\left(\frac{(1.08)^5 - 1}{1.08 - 1}\right) \\ &= 2933.30 \end{aligned}$$

Thereofre, the future value of the yearly investment over 5 years amounts to \$2933.30.

b) In order to determine the interest Scooter earned we need to determine the principal amount. This is the amount of money that Scooter invested out of his own pocket. Scooter invests \$500 each June 1 for 5 years. So,

$$Principal = P = 5 \times \$500 = \$2500$$

The future value, FV, is the amount after the 5 years was calculated in (a) to be \$2933.30. This means that the interest earned, I, is the difference,

$$\begin{aligned} Interest = I &= FV - P \\ &= 2933.30 - 2500 \\ &= 433.30 \end{aligned}$$

Therefore, the interest Scooter earns is \$433.30.

Summary

$$FV = PMT\left(\frac{(1+i)^n - 1}{i}\right) \tag{1}$$

where,

Pmt = value of each payment

r = interest rate, usually per year

i = $r/$ (number of compounding periods in a year) = interest rate per period

n = total number of compouninding periods over the entire investment period

FV = future value

Exercises

1. On your 18^{th} irthday you start making regular payments of \$20 every month into an RRSP that earns 3.25% per year compounded monthly. How much money will you have when you are 65 years old?

2. Calculuate the future value for the following annuities,

   a) \$650 invested monthly for 2 years into an account paying 3.9% / year compounded monthly.

   b) \$5000 invested semi-annually for 6 years into a mutual fund that historically pays 12% per year compounded semi-annually.

   c) \$2000 invested quarterly for 1 year into a fund that pays 2.25% per year compounded quartlerly.

3. Jane deposits \$100 on March 31, June 30, September 30 and December 31 every year for 20 years. The investment pays 4% / year compounded quarterly. How much is in the account when the last payment is made?

4. Say you want to retire in 30 years with \$1000000. You want to make equal monthly payments into an account that pays 10%/year compounded monthly. What monthly payment do you have make each month?