Transformations of Quadratics

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Vertex form of a quadratic

The **vertex from** of a quadratic is another way of writing a quadratic. The general **vertex form** looks as follows,

$$f(x) = a(bx - k)^2 + h$$

We can simplify this a bit by factoring out the b. When we do this we get,

$$\begin{aligned} f(x) &= ab\left(x - \frac{k}{b}\right) + h \\ &= A\,(x - K) + h \end{aligned}$$

where a is the vertical stretch or compression; K is the horizontal translation; h is the vertical translation. The *vertex* of the quadratic is given by (K, h). Something to note about the horizontal translation K, when $K > 0$ we have a translation to the *right*; when $K < 0$, we have a translation to the *left*.

Let's consider an example to get a feel for how and when best to use the vertex form of a quadratic.

Example

Sketch the following quadratic,

$$f(x) = 2(x - 1)^2 + 3$$

Solution

1. First, determine the vertex. In this case the vertex is $(1, 3)$.

2. Second, is the quadratic reflected in the x-axis? No. This means the quadratic opens upwards.

3. Third, what are the roots of x-intercepts? We need to solve $0 - f(x) = 2(x-1)^2 + 3$ to determine the x-intercepts.

$$\begin{aligned} 0 = f(x) &= 2(x-1)^2 + 3 \\ 0 &= 2(x^2 - 2x + 1) + 3 \\ &= 2x^2 - 4x + 2 + 3 \\ &= 2x^2 - 2x + 5 \end{aligned}$$

The quadratic does not factor easily so we will need to use the quadratic formula.

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{2 \pm \sqrt{2^2 - 4(2)(5)}}{2(2)} \\ &= \frac{2 \pm \sqrt{4 - 40}}{4} \\ &= \text{does not exist} \end{aligned}$$

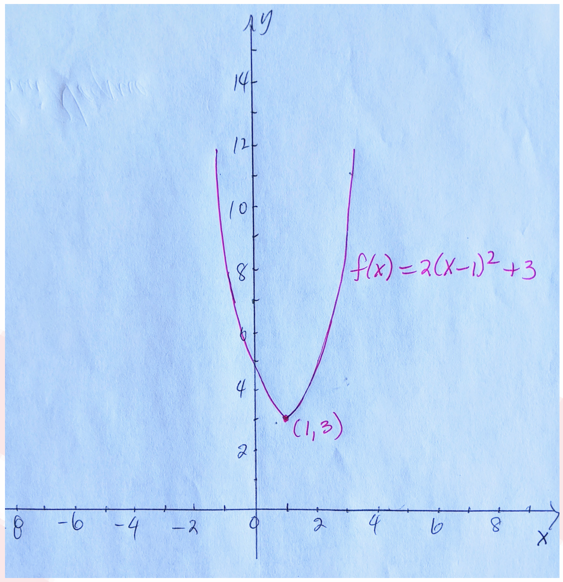
This means there are no x-intercepts.

4. Fourth, what is the y-intercept? To find the y-intercept we let $x = 0$ and then solve for y.

$$\begin{aligned} y &= 2(x-1)^2 + 3 \\ &= 2(0-1)^2 + 3 \\ &= 2 + 3 \\ &= 5 \end{aligned}$$

This means the y-intercept is $y = 5$.

5. Sketh the graph. We can use the information above to sketch the graph below.



When we're faced with a random quadratic equation and we know that working with the vertex form would be easier, how do we change our given quadratic into the vertex form? This is a good question and *completing the square* is the process that needs to be applied. Let's look at an example to see how this is done.

Example

Sketch the graph of the function,

$$f(x) = x^2 + 4x - 6.$$

Solution

1. First, complete the square. We do this so we can put the quadratic in the vertex form after which we can easily read off the vertex of the quadratic.

$$\begin{aligned} f(x) &= x^2 + 4x - 6 \\ &= x^2 + 4x + (4 - 4) - 6 \\ &= (x^2 + 4x + 4) - 10 \\ &= (x + 2)^2 - 10 \end{aligned}$$

2. Second, what is the vertex? Now we can read off the vertex from the vertex form fro step 1. The vertex is $(-2, -10)$.

3. Third, does the quadratic open up or down? The quadratic opens up since $a = 1 > 1$.

4. Fourth, are there any roots, x-intercepts? To find the x-intercepts we let $y = 0$ and then solve for x.

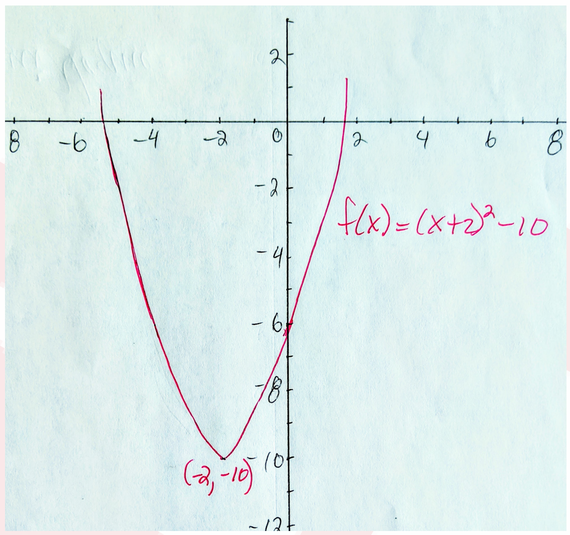
$$\begin{aligned} 0 &= x^2 + 4x - 6 \\ x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-4 \pm \sqrt{16 - 4(1)(-6)}}{2} \\ &= \frac{-4 \pm \sqrt{16 + 24}}{2} \\ &= \frac{-4 \pm \sqrt{40}}{2} \\ &= \frac{-4 \pm 2\sqrt{10}}{2} \\ &= -2 \pm \sqrt{10} \end{aligned}$$

Therefore, the x-intercepts are $x = -2 + \sqrt{10}$ and $x = -2 - \sqrt{10}$.

5. Fifth, what is the y-intercept? To find the y-intercept we let $x = 0$ and then solve for y.

$$y = 0^2 + (0) - 6 = -6$$

6. Final step is to sketch the graph fro the information above



Let's consider another different example.

Example

Sketch the graph of the function,

$$f(x) = -x^2 + 2x + 3$$

Solution

1. Complete the square so we can find the vertex of the quadratic,

parabola.

$$\begin{aligned} f(x) &= -x^2 + 2x + 3 \\ &= -(x^2 - 2x) + 3 \\ &= -(x^2 - 2x + 1 - 1) + 3 \\ &= -(x^2 - 2x + 1) + 1 + 3 \\ &= -(x^2 - 2x + 1) + 4 \\ &= -(x - 1)(x - 1) + 4 \\ &= -(x - 1)^2 + 4 \end{aligned}$$

The vertex form of the quadratic is,

$$f(x) = -(x - 1)^2 + 4$$

2. The vertex is $(1, 4)$.
3. The quadratic opens downwards because $a = -1 < 0$.
4. The roots, or x-intercepts are found by letting $y = 0$ and solving for x.

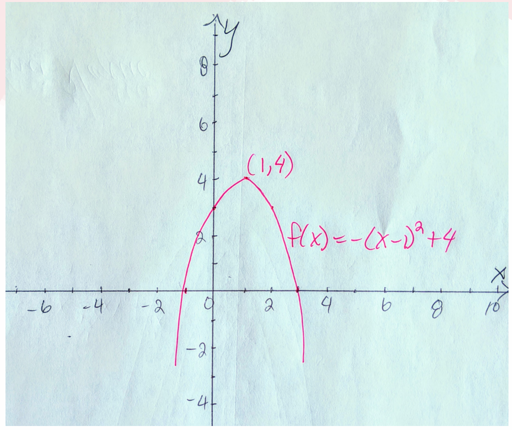
$$\begin{aligned} 0 &= -x^2 + 2x + 3 \\ &= x^2 - 2x - 3 \\ &= (x - 3)(x + 1) \end{aligned}$$

From here we see that the roots are $x = 3$ and $x = -1$.

5. The y-intercept is found by letting $x = 0$ and solving for y.

$$y = -0^2 + 2(0) + 3 = 3$$

6. Finally, sketch the graph.



When the quadratic is written in vertex form, we can easily read off the vertex of the quadratic. Given the **vertex form** of a quadratic,

$$y = a(b(x-k))^2 + h$$

then we have,

1. Vertex $= (k, h)$
2. If $a < 0$ then the quadratic opens downwards and (k, h) is a maximum value. If $a > 0$ then the quadratic opens upwards and (k, h) is a minimum.
3. If $b < 0$ then the quadratic opens towards the left; if $b > 0$ then the quadratic opens towards the right.
4. We can find the x-intercepts or roots using the quadratic formula.
5. We can find the y-intercept by letting $x = 0$.
6. The axis of symmetry is a vertical line $x = x_0$ where x_0 is the value midway between the two x-intercepts. For example, is x_1

and x_2 are the x-intercepts then,

$$x_0 \;=\; \frac{x_1 + x_2}{2}$$

is the *equation of the line of symmetry.*

Exercises

Sketch each quadratic below. What are the coordinates of the vertex for each?

1. $y = 2(x+2)^2 - 3$

2. $y = \frac{1}{4}x^2$

3. $y = -x^2 + 4$

4. $y = 2(x-3)^2 + 2$

5. $y = -\frac{1}{2}(x+1)^2 - 3$