Graphing Rational Functions

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2020

Rational Function

A rational number is a number that can be written as fraction. e.g.

$$\frac{4}{3}, \frac{6}{17}, -\frac{2}{3}$$

A *rational function* is a function that is written as a fraction of functions. In particular, a fraction of polynomials.

Rational Function A *rational function* hs the form $h(x) = \frac{f(x)}{g(x)}$ where $f(x)$ and $g(x)$ are polynomilas. The domain of a rational function is all real numbers except x for which $g(x) = 0$. That is,

$$\text{Domain} = \{x \in \mathbb{R} | g(x) \neq 0\}$$

The zeros of $h(x)$ are the zeros of $f(x)$ if $h(x)$ is in simplified form.

Graphing Rational Functions

The main features to consider when graphing a rational function are,

1. x-intercepts
2. y-intercepts
3. vertical asymptotes
4. horizontal asymptotes
5. domain
6. range

in any order that is most convenient for the function given.

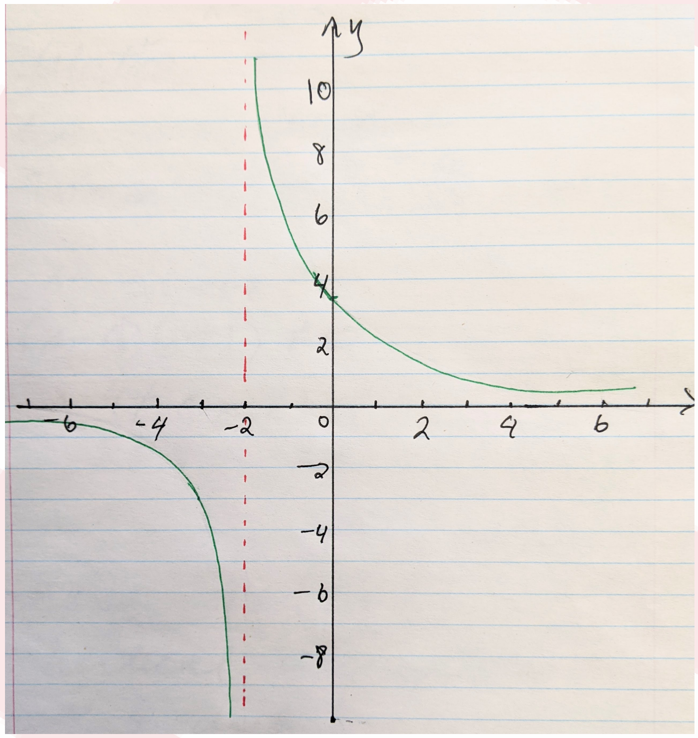
Example Graph $f(x) = \frac{7}{x+2}$

Solution

1. Domain $= \{x \in \mathbb{R} | x \neq -2\}$ Since we cannot divide by zero, $x + 2 \neq 0$ or $x \neq -2$.
2. Range $= \{y | y \in \mathbb{R}\}$
3. Veritcal asymptote: $x = -2$
4. Horizontal asymptote: $y = 0$
5. y-intercept:
$$f(0) = \frac{7}{2}$$
Therefore, the y-intercept is $y = \frac{7}{2}$.
6. x-intercept: There is no x-intercept since y=0 is a horizontal asymptote.

Next we will create table to help us determine on what intervals the function is negative and positive.

	$x < -2$	x $x > -2$
x+2	-	+
$\frac{7}{x+2}$	-	+



Example Graph $f(x) = \frac{4}{x^2-3x-4}$.

Solution

1. Domain $\{x \in \mathbb{R} | x^2 - 3x - 4 \neq 0\}$

$$\begin{aligned} x^2 - 3x - 4 &\neq 0 \\ (x-4)(x+1) &\neq 0 \\ x - 4 \neq - \quad &\text{or} \quad x + 1 \neq 0 \\ x \neq 4 \quad &\text{or} \quad x \neq -1 \end{aligned}$$

   Therefore, Domain $\{x \in \mathbb{R} | x \neq 4 \text{ or } x \neq -1\}$

2. Range $= \{y | y \in \mathbb{R}\}$

3. Vertical Asymptotes: $x = 4$ and $x = -1$.

4. Horizontal Asymptotes: $y = 0$

$$\begin{aligned}\lim_{x\to+\infty} f(x) &= 0^+ \text{ and} \\ \lim_{x\to-\infty} f(x) &= 0\end{aligned}$$

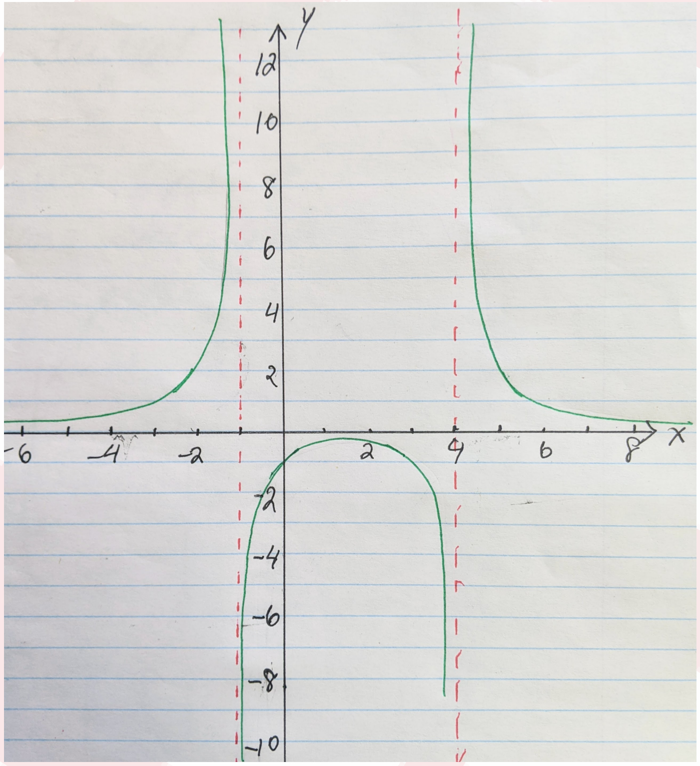
5. y-intercept:

$$f(0) = \frac{4}{-4} = -1$$

6. x-intercept: None since y=0 is a horizontal asymptote.

7. This table below helps us deermine when the function is positive or negative i.e. lies above the x-axis or below the x-axis, respectively.

	$x < -1$	$-1 < x < 4$	$4 < x$
$x - 4$	-	-	+
$x + 1$	-	+	+
$f(x) = \frac{4}{(x-4)(x+1)}$	+	-	+



Example Graph $\frac{x+2}{3x-2}$

Solution

1. Domain $= \{x \in \mathbb{R} | 3x - 2 \neq 0\}$

$$\begin{aligned} 3x - 2 &\neq 0 \\ 3x &\neq 2 \\ x &\neq 2/3 \end{aligned}$$

   Therefore, Domain $= \{x \in \mathbb{R} | x \neq \frac{2}{3}\}$.

2. Vertical Asymptotes: $x = \frac{2}{3}$

3. Horizontal Asymptotes:

$$\begin{aligned} f(x) &= \frac{x+2}{3x-2} \\ &= \frac{\frac{x}{x} = \frac{2}{x}}{\frac{3x}{x} - \frac{2}{x}} \\ &= \frac{1+\frac{2}{x}}{3-\frac{2}{x}} \end{aligned}$$

As $x \to +\infty$, $\frac{2}{x} \to 0^{=}$ and as $x \to -\infty$, $\frac{2}{x} \to 0^{-}$. Therefore, as $x \to +\infty$ or $x \to -\infty$ then $f(x) \to \frac{1}{3}$. Therefore, $y = \frac{1}{3}$ is a horizontal asymptote.

4. Range $= \{y \in \mathbb{R} | y \neq \frac{1}{3}\}$

5. x-intercept:

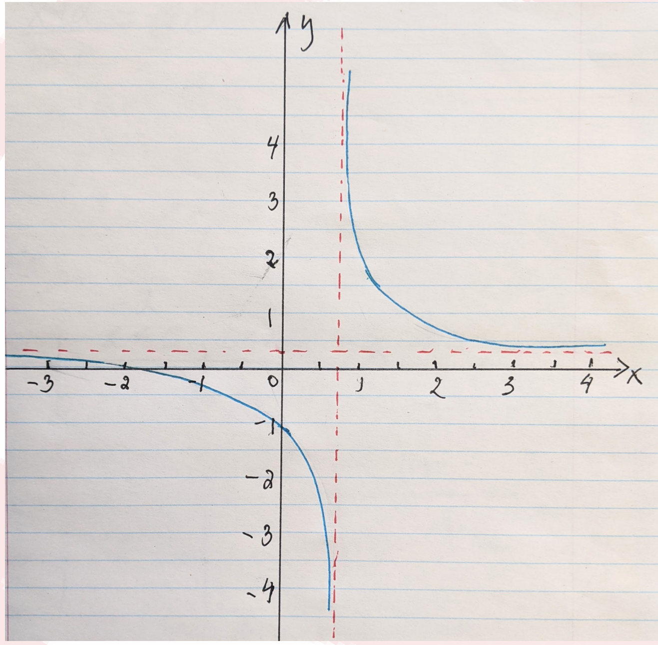
$$\begin{aligned} f(x) &= \frac{x+2}{3x-2} = 0 \\ \Rightarrow x+2 &= 0 \\ x &= -2 \end{aligned}$$

6. y-intercept:

$$f(x) = \frac{2}{-2} = -1$$

7. The table below will help us determine the intervals on which the function is positive and negatives

	$x < 2$	$0 < x < 2/3$	$2/3 < x$
$x-2$	-	+	+
$3x-2$	-	-	+
$\frac{x+2}{3x-2}$	+	-	+



Not all asymptotes are vertical or horizontal. An asymptote can also have the form of any line with any slope and y-intercept. This means an asymptote can have the form $y = mx + b$. This type of asymptote is called an *oblique asymptote*.

Oblique Asymptotes

The line $y = mx + b$ is an oblique or slanted asymptote to the graph of a rational function $h(x)$ if the vertical distance between the curve $y = h(x)$ and the line $y = mx + b$ approaches 0. In other words, the difference between $h(x)$ and $y = mx + b$ approaches 0 as x incrases or decrases without bound. So,

$$\lim_{x} \to \pm\infty(h(x) - (mx + b)) = 0$$

Question: How do you know if a rational function has an oblique asymptote?

Solution: A rational function $h(x) = \frac{f(x)}{g(x)}$ has an oblique asymptote if the degree of $f(x)$ = degree of $g(x) + 1$.

Example Does $f(x) = \frac{x^2-x-6}{x-2}$ have an oblique asymptote? If so, what is it?

Solution: The degree of the numerator, $x^2 - x - 6$ is 2 and the degree of the denominator, $x - 2$, is 1. So yes, $f(x)$ has an oblique asymptote because

$$degree(x^2 - x - 6) = degree(x - 2) + 1$$

Let's find the oblique asymptote.

$$\begin{array}{r|l} & x + 1 \\ \hline x-2 & x^2 - x - 6 \\ & -(x^2 - 2x) \\ \hline & x - 6 \\ & -(x - 2) \\ \hline & -4 \end{array}$$

Therefore, $f(x) = x + 1 - \frac{4}{x-2}$. As $x \to \pm\infty$, $f(x) \to x + 1$. Therefore, $y = x + 1$ is the oblique asymptote.

Example Graph $f(x) = \frac{x^2-x-6}{x-2}$

Solution

1. Domain $= \{x \in \mathbb{R} | x - 2 \neq 0\} = \{x \in \mathbb{R} | x \neq 2\}$

2. y-intercept

$$f(0) = \frac{0 - 0 - 6}{0 - 2} = 4$$

3. x-intercept

$$\begin{aligned} 0 &= \frac{x^2-x-6}{x-2} = \frac{(x-3)(x+2)}{x-2} \\ 0 &= x-2 \text{ or } 0 = x+2 \\ 3 &= x \text{ or } -2 = x \end{aligned}$$

4. Vertical Asymptote: x=2

5. Horizontal Asymptote:

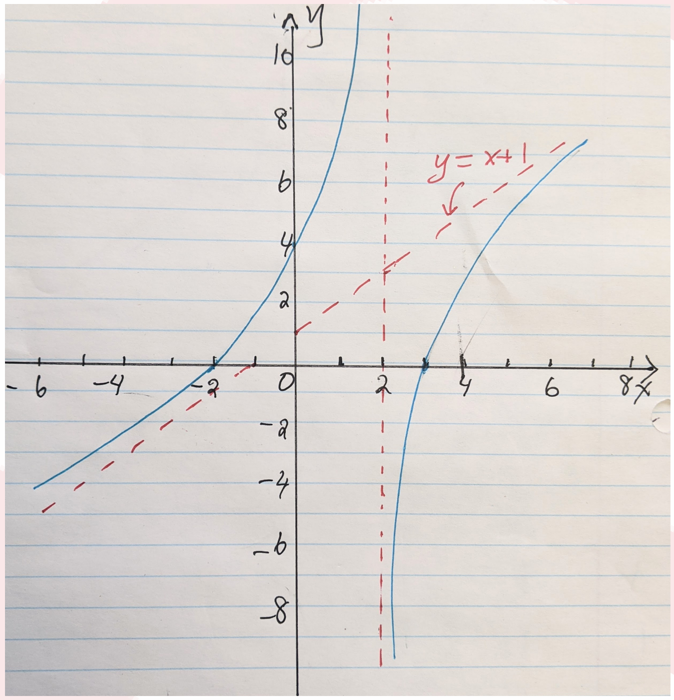
$$\begin{aligned} f(x) &= \frac{x^2-x-6}{x-2} \\ &= \frac{\frac{x^2}{x^2}-\frac{x}{x^2}-\frac{6}{x^2}}{\frac{x}{x^2}-\frac{2}{x^2}} \\ &= \frac{1-\frac{1}{x}-\frac{6}{x^2}}{\frac{1}{x}-\frac{2}{x^2}} \end{aligned}$$

As $x \to \pm\infty$ then $\frac{1}{x} \to 0$, $\frac{6}{x^2} \to 0$ and $-\frac{2}{x^2} \to 0$ but $\frac{1}{x} - \frac{2}{x^2} \to 0$. So, $f(x) \to \frac{1}{0}$ which is undefined. Therefore, there is no horizontal asymptote.

6. Range $= \{y \in \mathbb{R} | y \neq x+1\}$

7. The table below will help us determine the intervals on which the function is positive and negative.

	$x < -2$	$-2 < x < -1$	$-1 < x < 2$	$2 < x < 3$	$3 < 2$
$x-3$	-	-	-	-	+
$x+2$	-	+	+	+	+
$x-2$	-	-	-	+	+
$\frac{(x-3)(x+2)}{x-2}$	-	+	+	-	+


y
x
y = x+1
10
8
6
4
2
-2
-4
-6
-8
-6
-4
-2
0
2
4
6
8

Exercises

For the following functions find,

(i) The x and y intercepts

(ii) Domain and range.

(iii) The equations of the vertical, horizontal and oblique asymptotes, if any.

(iv) Positive and negative intervals of the function.

(v) Graph the function.

(a)
$$f(x) = \frac{2+x}{x-7}$$

(b)
$$f(x) = \frac{5x^2 - 11x + 2}{x}$$

(c)
$$f(x) = \frac{x^2 + x - 6}{x+2}$$

(d)
$$f(x) = \frac{x^2 - 9}{x^3 + 4x^2 - x - 4}$$

(e)
$$f(x) = \frac{x^2 - 4}{x}$$

(f)
$$f(x) = \frac{2x^2 - 5x}{x^2 - 1}$$

(g)

$$f(x) = \frac{x^2 + 4x + 3}{x - 2}$$

(h)

$$f(x) = \frac{x^2}{x^3 - 2x^2 - x + 2}$$