Zeros or Roots of Polynomials - 1

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Zeros and Roots of Polynomials

The zeros or roots of a polynomial are the values of the independent variable usually x that result in the polynomial having a value of zero. These are the points on the graph of the polynomial that intersect the x-axis. Let's start by looking at *quadratics* and see how many possible zeros or roots a quadratic can have and what the graph of that quadratic will look like.

Quadratic

What is a quadratic? A quadratic is a polynomial with degree of 2. The general form of a quadratic is given by,

$$f(x) = ax^2 + bx + c, \quad \text{where } a, b, c \in \mathbb{R}$$

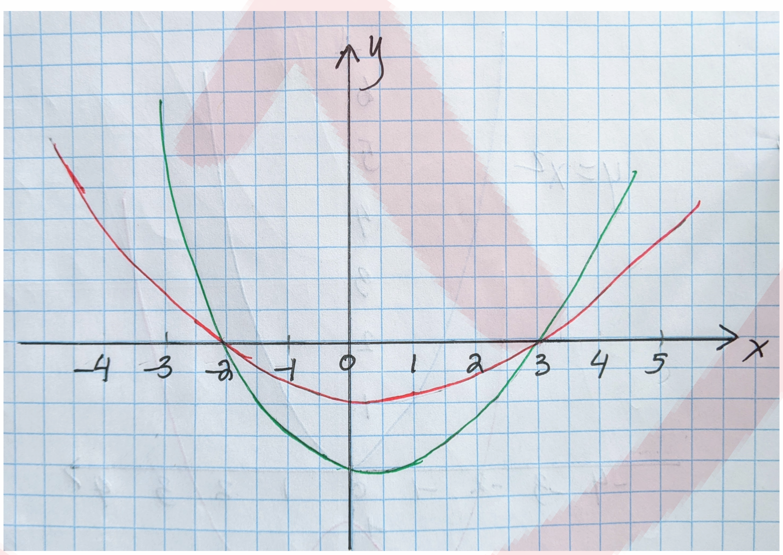
Let's start by considering a quadratic with 2 real roots or zeros.

Two real roots

Let's consider the following example of a quadratic with 2 real roots.

$$\begin{aligned} & f(x) = a(x+2)(x-3) \\ & \text{zeros of } f: -2,\ 3 \\ & \text{roots of } f(x) = 0: -2, \quad 3 \end{aligned}$$

What does the graph of this quadratic look like? Let's assume $a > 0$.



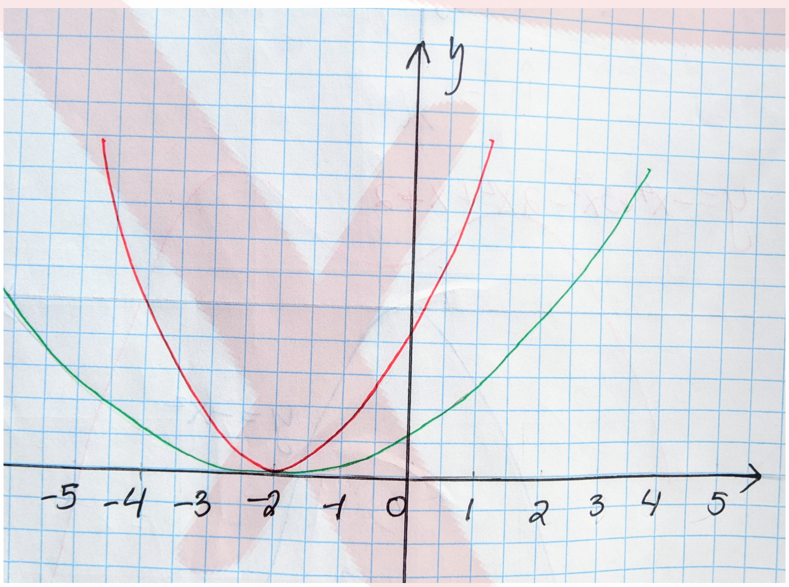
When $a > 1$ we have a thinner, stretched out curve; when $a < 1$ we have a flatter curve. When $a < 0$ then our curve opens downwards. Since we have two real roots, the curve crosses the x-axis twice.

One real root

Let's consider the following example of a quadratic with only 1 real root.

$$\begin{aligned} f(x) &= a(x+2)^2 \\ \text{zeros of } f &: -2 \\ \text{roots of } f(x) = 0 &: -2 \end{aligned}$$

What does the graph of this quadratic look like? Again, let's assume $a > 0$.



When $a > 1$ we have a thinner, stretched out curve; when $a < 1$ we have a flatter, compressed curve; when $a < 0$, our curve opens downwards. Since we have only one real root, the curuve touches the x-axis once.

Cubic

What is a cubic? A cubic is a polynomial of degree 3. Recall that the degree of a polynomial is the value of the highest power in the polynomial. A general cubic is given by,

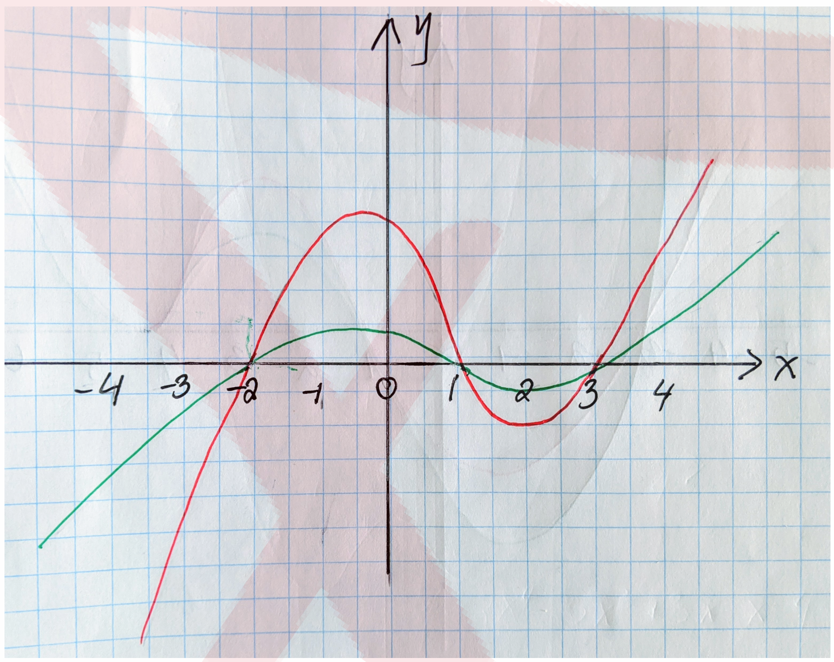
$$f(x) = ax^3 + bx^2 + cx + d$$

Three real roots

Let's consider the following example of a cubic with 3 real roots.

$$f(x) = a(x+2)(x-1)(x-3)$$
$$\text{zeros of } f : -2,\ 1,\ 3$$
$$\text{roots of } f(x) = 0 : -2,\ 1,\ 3$$

What does the graph of this type of cubic look like? Again, we'll assume $a > 0$ for our sketch.



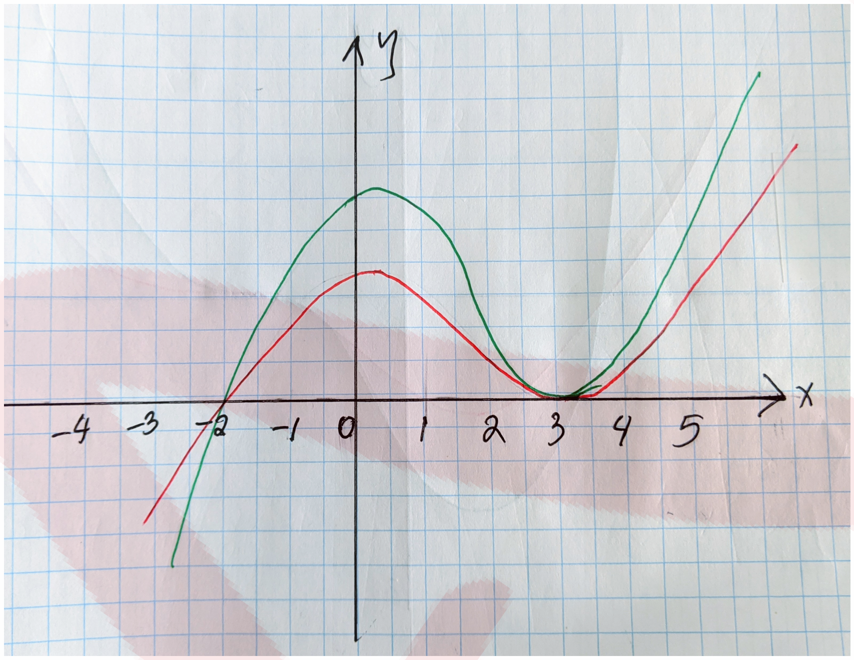
Since we have 3 real roots, the graph crosses the x-axis 3 times and at the zeros or roots,-2, 1 and 3.

Two real roots

Let's consider the following example of a cubic with 2 real roots.

$$f(x) = a(x+2)(x-3)^2$$
$$\text{zeros of } f : -2,\ 3$$
$$\text{roots of } f(x) = 0 : -2,\ 3$$

What does the graph of this cubic look like? Assume that $a > 0$.



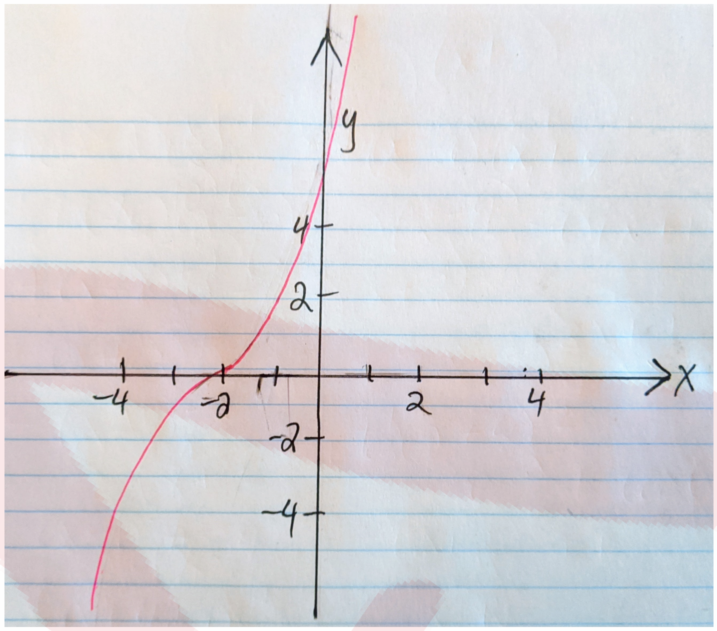
Since we have 2 real roots, the graph crosses or touches the x-axis 2 times. Notice that the factor $(x-3)^2$ contributes the zero $x=3$ but is also squared. This zero is the point at which the graph just touches the x-axis.

One real roots

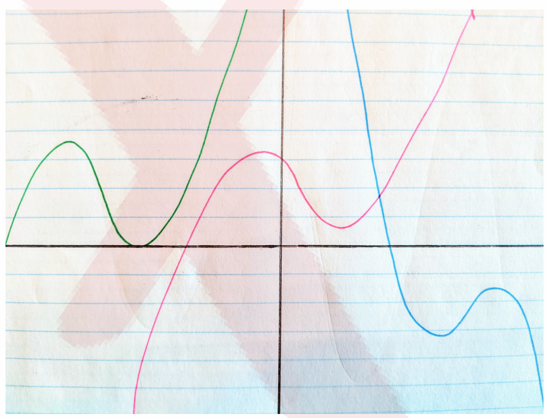
Let's consider the following example of a cubic with 1 real root.

$$f(x) = a(x+2)^3$$
$$\text{zeros of } f : -2$$
$$\text{roots of } f(x) = 0 : -2$$

What does the graph of this cubic look like? Assume $a > 0$. Recall the parent function $f(x) = x^3$? We know what the graph of this function looks like. The graph of $f(x) = a(x+2)^3$ is this parent function transformted. What are the transformations? The first is the multiplication of $a > 0$ which is a stretch or compression depending on whether $a > 1$ or $0 < a < 1$. The second transformation is the translation horizontally to the left 2 units. So the shape of this graph is similar to x^3 except that the "centre" is at $x = -2$ where the graph crosses the x-axis.



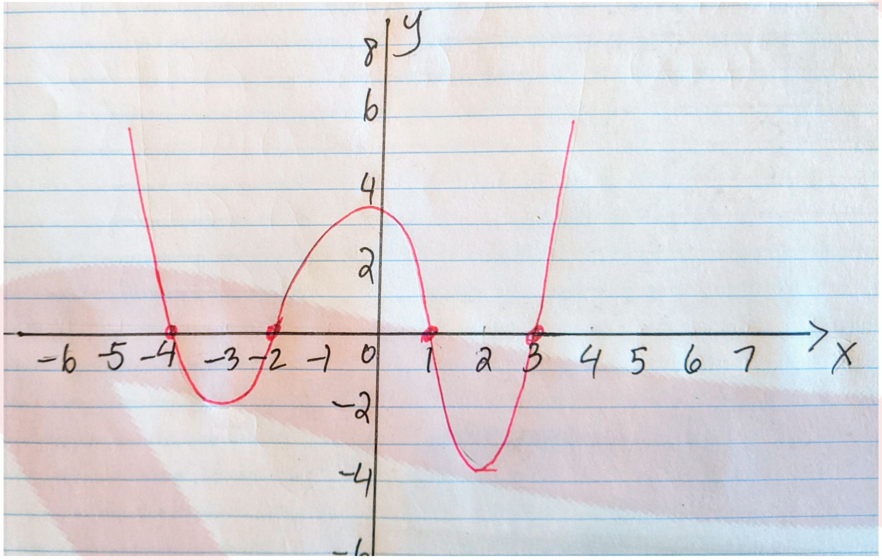
Here are some other examples of what a cubic may look like with one root



Quartic

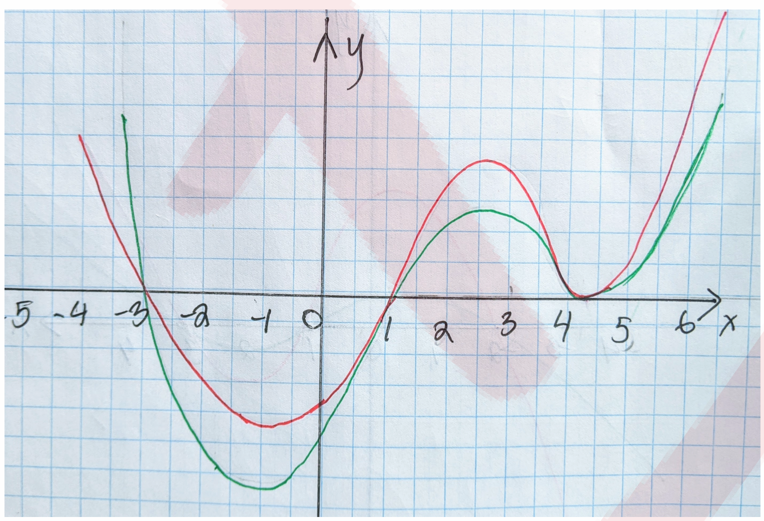
Four real roots

$$f(x) = a(x+2)(x-1)(x-3)(x+4)$$
$$\text{zeros of } f: -2,\ 1,\ 3,\ -4$$
$$\text{roots of } f(x)=0: -2,\ 1,\ 3,\ -4$$



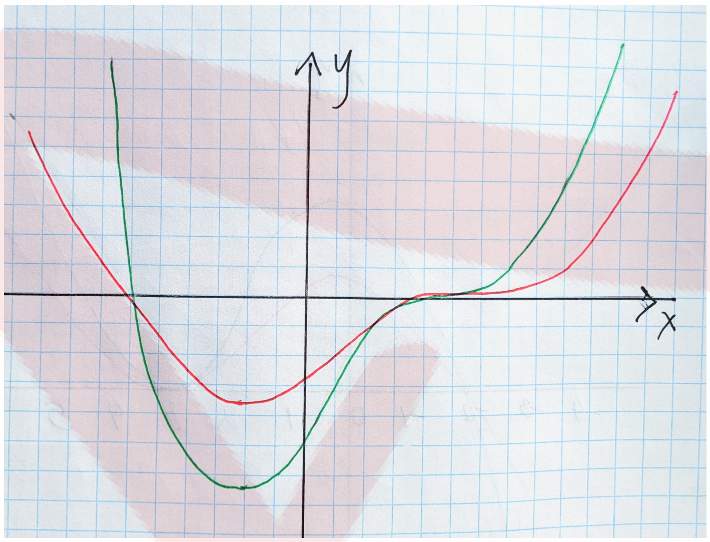
Three real roots

$$f(x) = a(x+2)(x-1)(x-3)^2$$
$$\text{zeros of } f : -2,\ 1,\ 3$$
$$\text{roots of } f(x) = 0 : -2,\ 1,\ 3$$



Two real roots

$$f(x) = a(x+3)(x-1)^3$$
$$\text{zeros of } f: -3,\ 1$$
$$\text{roots of } f(x)=0: -3,\ 1$$



Note: For the above examples, the diagrams are just one representation of that particular case. There may be other diagrams representing each situation.

Exercises

1. Which funtions are polynomial functions?

   a) $f(x) = x^{10} + x^4 + 4$

   b) $f(x) = \frac{x^2+2x+1}{x+1}$

   c) $f(x) = e^{-x} + \cos\left(\frac{x}{2}\right)$

   d) $f(x) = \frac{x}{2}$

   e) $f(x) = x^{100} + 6$

2. Draw a sketch of the following functions.

   a) $y = -2(x+3)^2 - 4$

   b) $y = (x+1)^3$

   c) $y = -(x+2)^3 - 3$

   d) $y = (x-4)^4$

   e) $y = -(x+2)^2 - 1$

   f) $y = x^2 - 1$

g) $y = -2(x+1)(x-3)(x-5)$
h) $y = 3(x+4)^2$
i) $y = x^3 - 2$
j) $y = (x-10)(x+1)$
k) $y = -x^4$
l) $y = (x^2 + 5x + 6)^2$
m) $y = -2x^2 + 4$