Related Rates 2

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Related Rates

What is a related rate? Ths best way to explain this is through an example.

Example

Let's consider oil spilt from a tanker. The spilt oil spread in a circle whose area increases at a constant rate of $6km^2/h$. How fast is the radius of the spill increasing when the area is $9\pi km^2$?

Solution First, what is the formula for the area of a circle?

$$\text{area of a circle } = A = \pi r^2$$

Let's differentiate both sides of this equation w.r.t. t.

$$\frac{dA}{dt} = \pi 2r\frac{r}{t} \text{ This is our related rate equation}$$

We are given that $\frac{dA}{dt} = 6km^2/t$ and we want to find $\frac{dr}{dt}$ when $A = 9\pi km^2$. Since there is no A in the related rate equation,

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

but there is an r. We can find the value of r when $A = 9\pi km^2$.

Step 1 Find the radius r when $A = 9\pi km^2$.

$$\begin{aligned} A &= \pi r^2 \\ \frac{9\pi}{\pi} &= \frac{\pi r^2}{\pi} \\ 9 &= r^2 \\ 3 &= r \end{aligned}$$

Therefore, when $r = 3km$, the area is $A = 9\pi km^2$.

Step 2 Next, we need to find the rates of change of the radius when the radius is $r = 3km$ or when $A = 9\pi km^2$.

$$\begin{aligned}\frac{dA}{dt} &= 2\pi r\frac{dr}{dt}\\ 6 &= 2\pi(3)\frac{dr}{dt}\\ \frac{6}{6\pi} &= \frac{6\pi}{6\pi}\frac{dr}{dt}\\ \frac{1}{\pi}km/h &= \frac{dr}{dt}\end{aligned}$$

Therefore, the radius of the spill is increasing at a rate of $1/\pi km/h$.

Another example.

Example

A raindrop falls into a puddle. Riplles spread out into concentric circles from where the raindrop hits. The radii of the circles grow at the rate of $2cm/s$.

a) Find the rate of increase of the circumfrence of one circle.

b) Find the rate of increase of the area of the circle that has an area of $121\pi cm^2$.

Solution

Step 1 Let's write some relevant formulas down. The example asks for circumfrence and area. What are the formulas for the circumfrence and area of a circle?

$$\begin{aligned}C &= 2\pi r, \quad \text{is the formula for the circumfrence of a circle}\\ A &= \pi r^2, \quad \text{is the formula for the area of a circle}\end{aligned}$$

paragraphStep 2 Let's take some derivatives. Since the examples gives the rate of increase of the radius, asks about the rate of increase of the circumfrence and the rate of increase of the area, let's implicitly differentiate w.r.t. t our formulas from Step 1.

$$\begin{aligned}\frac{dC}{dt} &= 2\pi\frac{dr}{dt}\\ \frac{dA}{dt} &= 2\pi r\frac{dr}{dt}\\ \therefore & \quad \frac{dC}{dt}, \frac{dA}{dt} \text{ and } \frac{dr}{dt}, \text{ are our rates of change.}\end{aligned}$$

Step 3 What information are we given? We are given taht the radii increase at $2cm/s$ or $\frac{dr}{dt} = 2cm/s$.

Step 4 Let's start answering each equestion.

a) The rate of increase of the circumference is given by

$$\frac{dC}{dt} = 2\pi\frac{dr}{dt} = 2\pi(2cm/s) = 4\pi cm/s$$

b) Next, we want to find the rate of change of the area when the area is $121\pi cm^2$. First, we notice that the rate of change of area equation doesn't include the area but it does include the radius. So we need to find the value of the radius when the area is $121\pi cm^2$.

$$\begin{aligned} A &= \pi r^2 \\ \frac{121\pi}{\pi} &= \frac{\pi r^2}{\pi} \\ 121 &= r^2 \\ 11cm &= r \end{aligned}$$

This means that the radius when the area of the circle is $121cm^2$ is $r = 11cm$. We will use this value of the radius in the rate of change of area equation.

$$\begin{aligned} \frac{dA}{dt} &= 2\pi r\frac{dr}{dt} \\ &= 2\pi(11cm)(2cm/s) \\ \frac{dA}{dt} &= 44\pi cm^2/s \end{aligned}$$

Therefore, the rate of change of the area when the area is $121cm^2$ is $44\pi cm^2/s$.

Procedure for solving related rate problems

1. Sketch and label quantities when possible.
2. Introduce variables to represent quantities that change.
3. Identify quantities to be found,
4. Write down equations that relate the variables.
5. Implicitly differentiate both sides of the equation w.r.t. t.

6. Substitute in all known values for variables and related rates.
7. Solve the equations for required rate of change.
8. Writ a concluding statement answering the relevant question.

Exercises

1. Express the following stateents in symbols:
   a) The areas, A, of a circle is increasing at a rate of $4m^2/s$.
   b) The surface area, S, of sphere is decreasing at a rate of 3 m^2/min.
   c) After travelling for 15 minutes, the speed of a car is $70km/h$.
   d) The x and y coorindates of a point are changing at equal rates.
   e) The head of a short-distance radar dish is revolving at three revolutions per minute.
2. The side of a square is increasing at a rate of 5 cm/s. At what rate is the area changing when the side is 10 cm long? At what rate is the perimeter changing?
3. Each edge of a cube is expanding at ar ate of 4 cm/s.
   a) How fast is the volume changing when each edge is 5 cm?
   b) At what rate is the surface area changing when each edge is 7 cm?
4. One side of a rectangle increases at 2 cm/s, while the other side decreases at 3 cm/s. How rfast is the area of the rectangle changing when the first die equals 20 cm an the second side equals 50 cm?
5. The area of a circle ids decreasinga t the rate of 5 m^2/s when its radius is 3m.
   a) At what rate is the ardius decreasing at that moment?
   b) At what rate is the diamter decreasing at that moment?