Related Rates

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What is a related rate? Ths best way to explain this is through an example.

Example

Let's consider oil spilt from a tanker. The spilt oil spread in a circle whose area increases at a constant rate of $6km^2/h$. How fast is the radius of the spill increasing when the area is $9\pi km^2$?

Solution First, what is the formula for the area of a circle?

$$\text{area of a circle } = A = \pi r^2$$

Let's differentiate both sides of this equation w.r.t. t.

$$\frac{dA}{dt} = \pi 2r\frac{r}{t} \text{ This is our related rate equation}$$

We are given that $\frac{dA}{dt} = 6km^2/t$ and we want to find $\frac{dr}{dt}$ when $A = 9\pi km^2$. Since there is no A in the related rate equation,

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

but there is an r. We can find the value of r when $A = 9\pi km^2$.

Step 1 Find the radius r when $A = 9\pi km^2$.

$$\begin{aligned} A &= \pi r^2 \\ \frac{9\pi}{\pi} &= \frac{\pi r^2}{\pi} \\ 9 &= r^2 \\ 3 &= r \end{aligned}$$

Therefore, when $r = 3km$, the area is $A = 9\pi km^2$.

Step 2 Next, we need to find the rates of change of the radius when the radius is $r = 3km$ or when $A = 9\pi km^2$.

$$\begin{aligned}\frac{dA}{dt} &= 2\pi r\frac{dr}{dt}\\ 6 &= 2\pi(3)\frac{dr}{dt}\\ \frac{6}{6\pi} &= \frac{6\pi}{6\pi}\frac{dr}{dt}\\ \frac{1}{\pi}km/h &= \frac{dr}{dt}\end{aligned}$$

Therefore, the radius of the spill is increasing at a rate of $1/\pi km/h$.

Another example.

Example

A raindrop falls into a puddle. Riplles spread out into concentric circles from where the raindrop hits. The radii of the circles grow at the rate of $2cm/s$.

a) Find the rate of increase of the circumfrence of one circle.

b) Find the rate of increase of the area of the circle that has an area of $121\pi cm^2$.

Solution

Step 1 Let's write some relevant formulas down. The example asks for circumfrence and area. What are the formulas for the circumfrence and area of a circle?

$$\begin{aligned}C &= 2\pi r, \quad \text{is the formula for the circumfrence of a circle}\\ A &= \pi r^2, \quad \text{is the formula for the area of a circle}\end{aligned}$$

paragraphStep 2 Let's take some derivatives. Since the examples gives the rate of increase of the radius, asks about the rate of increase of the circumfrence and the rate of increase of the area, let's implicitly differentiate w.r.t. t our formulas from Step 1.

$$\begin{aligned}\frac{dC}{dt} &= 2\pi\frac{dr}{dt}\\ \frac{dA}{dt} &= 2\pi r\frac{dr}{dt}\\ \therefore & \quad \frac{dC}{dt}, \frac{dA}{dt} \text{ and } \frac{dr}{dt}, \text{ are our rates of change.}\end{aligned}$$

Step 3 What information are we given? We are given taht the radii increase at $2cm/s$ or $\frac{dr}{dt} = 2cm/s$.

Step 4 Let's start answering each equestion.

a) The rate of increase of the circumference is given by

$$\frac{dC}{dt} = 2\pi\frac{dr}{dt} = 2\pi(2cm/s) = 4\pi cm/s$$

b) Next, we want to find the rate of change of the area when the area is $121\pi cm^2$. First, we notice that the rate of change of area equation doesn't include the area but it does include the radius. So we need to find the value of the radius when the area is $121\pi cm^2$.

$$\begin{aligned} A &= \pi r^2 \\ \frac{121\pi}{\pi} &= \frac{\pi r^2}{\pi} \\ 121 &= r^2 \\ 11cm &= r \end{aligned}$$

This means that the radius when the area of the circle is $121cm^2$ is $r = 11cm$. We will use this value of the radius in the rate of change of area equation.

$$\begin{aligned} \frac{dA}{dt} &= 2\pi r\frac{dr}{dt} \\ &= 2\pi(11cm)(2cm/s) \\ \frac{dA}{dt} &= 44\pi cm^2/s \end{aligned}$$

Therefore, the rate of change of the area when the area is $121cm^2$ is $44\pi cm^2/s$.

Procedure for solving related rate problems

1. Sketch and label quantities when possible.
2. Introduce variables to represent quantities that change.
3. Identify quantities to be found,
4. Write down equations that relate the variables.
5. Implicitly differentiate both sides of the equation w.r.t. t.

6. Substitute in all known values for variables and related rates.
7. Solve the equations for required rate of change.
8. Writ a concluding statement answering the relevant question.

Exercises

For each curve, find the equation of the tangent at the given point.

a) $x^2 + y^2 = 13$ at $(2, -3)$

b) $x^2 + 4y^2 = 100$ at $(-8, 3)$

c) $\frac{x^2}{25} - \frac{y^2}{36} = -1$ at $(5\sqrt{3}, -12)$

d) $\frac{x^2}{81} - \frac{5y^2}{162} = 100$ at $(-11, -4)$

e) $5x^2 + y^2 = 21$ at $(-2, -1)$