

Rational Functions and  
Indeterminant Limits -3

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## What is a rational function?

Any number that can be written as a fraction is rational. In a similar manner of definition, any function that can be written as a fraction of functions, is called a *rational function*.

If  $f(x)$  is a polynomial function then

$$\lim_{x \rightarrow a} f(x) = f(a)$$

However, sometimes when you take a limit of a rational function you end up with something like  $\frac{0}{0}$  which makes no sense. This is an indeterminate form. We can use different techniques to try and evaluate this kind of limits.

### Example

Let's consider an example. What is the following limit,

$$\lim_{x \rightarrow 0} \frac{\sqrt{x+1} - 1}{x}$$

**Solution:** This limit is of the  $\frac{0}{0}$  indeterminate form. Let's evaluate this limit.

$$\begin{aligned} & \lim_{x \rightarrow 0} \left( \frac{\sqrt{x+1} - 1}{x} \right) \left( \frac{\sqrt{x+1} + 1}{\sqrt{x+1} + 1} \right), \quad \text{Rationalize the numerator} \\ &= \lim_{x \rightarrow 0} \frac{x+1-1}{x(\sqrt{x+1}-1)} \\ &= \lim_{x \rightarrow 0} \frac{x}{x(\sqrt{x+1}+1)} \\ &= \lim_{x \rightarrow 0} \frac{1}{\sqrt{x+1}+1} \\ &= \frac{1}{1+1} \\ &= \frac{1}{2} \end{aligned}$$

### Example

Let's try another example. Evaluate the following limit,

$$\lim_{x \rightarrow 1} \frac{x-1}{\sqrt{x}-1}$$

Solution:

$$\begin{aligned}
 & \lim_{x \rightarrow 1} \frac{x-1}{\sqrt{x}-1} \\
 &= \lim_{x \rightarrow 1} \left( \frac{x-1}{\sqrt{x}-1} \right) \left( \frac{\sqrt{x}+1}{\sqrt{x}+1} \right) \\
 &= \lim_{x \rightarrow 1} \frac{(x-1)(\sqrt{x}+1)}{x-1}, \quad \text{Rationalize the denominator} \\
 &= \lim_{x \rightarrow 1} (\sqrt{x}+1) \\
 &= 2
 \end{aligned}$$

Another type of example of an indeterminate form is the following,

$$\lim_{x \rightarrow 0} \frac{(x+8)^{1/3} - 2}{x}$$

We'll use a *substitution* to evaluate the above limit. In particular, let  $u = (x+8)^{1/3}$ . Then,  $u^3 = x+8$  and  $u^3 - 8 = x$ . As  $x \rightarrow 0$  we have  $u \rightarrow 2$ . So our limit becomes,

$$\begin{aligned}
 & \lim_{u \rightarrow 2} \frac{u-2}{u^3-8} \\
 &= \lim_{u \rightarrow 2} \frac{u-2}{(u-2)(u^2+2u+4)} \\
 &= \lim_{u \rightarrow 2} \frac{1}{u^2+2u+4} \\
 &= \lim_{u \rightarrow 2} \frac{1}{4+4+4} \\
 &= \frac{1}{12}
 \end{aligned}$$

One more example.

$$\lim_{x \rightarrow 2} \frac{|x-2|}{x-2}$$

Since we're dealing with the absolute value function, we need to consider the limit from the left and right hand sides.

$$\lim_{x \rightarrow 2^-} \frac{|x-2|}{x-2} \quad \text{and} \quad \lim_{x \rightarrow 2^+} \frac{|x-2|}{x-2}$$

Let's consider the function,

$$f(x) = \frac{|x-2|}{x-2} = \begin{cases} \frac{x-2}{x-2} & \text{if } x > 2 \\ -\frac{(x-2)}{x-2} & \text{if } x < 2 \end{cases}$$

$$= \begin{cases} 1 & \text{if } x > 2 \\ -1 & \text{if } x < 2 \end{cases}$$

Therefore,

$$\lim_{x \rightarrow 2^-} \frac{|x-2|}{x-2} = -1 \text{ and } \lim_{x \rightarrow 2^+} \frac{|x-2|}{x-2} = 1.$$

Since,

$$\lim_{x \rightarrow 2^-} \frac{|x-2|}{x-2} \neq \lim_{x \rightarrow 2^+} \frac{|x-2|}{x-2}$$

the limit,

$$\lim_{x \rightarrow 2} \frac{|x-2|}{x-2}$$

does not exist. Let's consider a couple of more straight forward examples.

$$\begin{aligned} & \lim_{x \rightarrow -1} \frac{x^2 - 5x + 2}{2x^3 + 3x + 1} \\ &= \frac{\lim_{x \rightarrow -1} (x^2 - 5x + 2)}{\lim_{x \rightarrow -1} (2x^3 + 3x + 1)} \\ &= \frac{1 + 5 + 2}{-2 - 3 + 1} \\ &= \frac{8}{-4} \\ &= -2 \end{aligned}$$

The different ways to evaluate indeterminate forms of limits include,

1. direct substitution
2. factoring
3. rationalizing
4. change of variable
5. one-sided limits

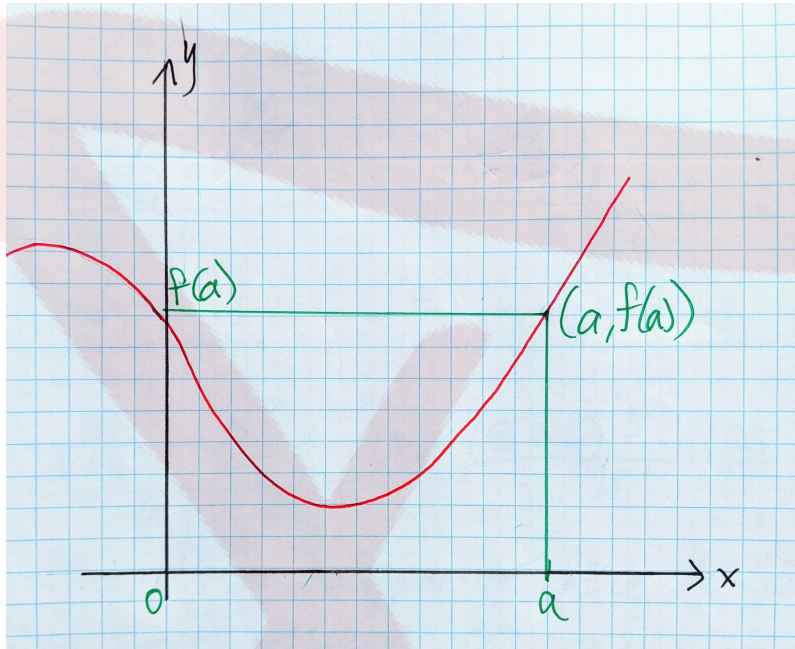
Now that we have a formal definition and understanding of the limit of a function, we can define *continuity of a function at a point*.

### Continuity Definition

The function  $f(x)$  is continuous at  $x = a$  if  $f(a)$  is defined and if

$$\lim_{x \rightarrow a} f(x) = f(a)$$

Otherwise,  $f(x)$  is discontinuous at  $x = a$ .



## Exercises

1. Evaluate the limits of the indeterminant quotients.

a)

$$\lim_{x \rightarrow 3} \frac{\sqrt{x+1} - 2}{x-3}$$

b)

$$\lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h}$$

c)

$$\lim_{x \rightarrow -1} \frac{x^2 + x}{x+1}$$

d)

$$\lim_{x \rightarrow 2} \frac{x^2 - x - 2}{|x-2|}$$

e)

$$\lim_{x \rightarrow 1} \left( \frac{1}{x-1} \right) \left( \frac{1}{x+3} - \frac{2}{3x+5} \right)$$

f)

$$\lim_{x \rightarrow 5/2} \frac{|2x-5|(x+1)}{(2x-5)}$$

g)

$$\lim_{x \rightarrow -2} \frac{(x+2)^3}{|x+2|}$$

h)

$$\lim_{x \rightarrow 0} \frac{\sqrt{x+1} - \sqrt{2x+1}}{\sqrt{3x+4} - \sqrt{2x+4}}$$