Rational Functions and Indeterminant Limits

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What is a rational function?

Any number that can be written as a fraction is rational. In a similar manner of definition, any function that can be written as a fraction of functions, is called a *rational function*.

If $f(x)$ is a polynomial function then

$$\lim_{x \to a} f(x) = f(a)$$

However, sometimes when you take a limit of a rational function you end up with something like $\frac{0}{0}$ which makes no sense. This is an indeterminant form. We can use different techniques to try and evaluate this kind of limits.

Example

Let's consider an example. What is the following limit,

$$\lim_{x \to 0} \frac{\sqrt{x+1}-1}{x}$$

Solution: This limit is of the $\frac{0}{0}$ indeterminant form. Let's evaluate this limit.

$$\begin{aligned}
& \lim_{x \to 0} \left(\frac{\sqrt{x+1}-1}{x} \right) \left(\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1} \right), \quad \text{Rationalize the numerator} \\
= & \lim_{x \to 0} \frac{x+1-1}{x(\sqrt{x+1}-1)} \\
= & \lim_{x \to 0} \frac{x}{x(\sqrt{x+1}+1)} \\
= & \lim_{x \to 0} \frac{1}{\sqrt{x+1}+1} \\
= & \frac{1}{1+1} \\
= & \frac{1}{2}
\end{aligned}$$

Example

Let's try another example. Evaluate the following limit,

$$\lim_{x \to 1} \frac{x-1}{\sqrt{x}-1}$$

Solution:

$$\begin{aligned}
& \lim_{x\to 1}\frac{x-1}{\sqrt{x}-1} \\
= & \lim_{x\to 1}\left(\frac{x-1}{\sqrt{x}-1}\right)\left(\frac{\sqrt{x}+1}{\sqrt{x}+1}\right) \\
= & \lim_{x\to 1}\frac{(x-1)(\sqrt{x}+1)}{x-1}, \quad \text{Rationalize the denominator} \\
= & \lim_{x\to 1}(\sqrt{x}+1) \\
= & \ 2
\end{aligned}$$

Another type of example of an indeterminate form is the following,

$$\lim_{x\to 0}\frac{(x+8)^{1/3}-2}{x}$$

We'll use a *substitution* to evaluate the above limit. In particular, let $u=(x+8)^{1/3}$. Then, $u^3=x+8$ and $u^3-8=x$. As $x\to 0$ we have $u\to 2$. So our limit becomes,

$$\begin{aligned}
& \lim_{u\to 2}\frac{u-2}{u^3-8} \\
= & \lim_{u\to 2}\frac{u-2}{(u-2)(u^2+2u+4)} \\
= & \lim_{u\to 2}\frac{1}{u^2+2u+4} \\
= & \lim_{u\to 2}\frac{1}{4+4+4} \\
= & \ \frac{1}{12}
\end{aligned}$$

One more example.

$$\lim_{x\to 2}\frac{|x-2|}{x-2}$$

Since we're dealing with the absolute value function, we need to consider the limit from he left and right hand sides.

$$\lim_{x\to 2^-}\frac{|x-2|}{x-2} \text{ and } \lim_{x\to 2^+}\frac{|x-2|}{x-2}$$

Let's consider the function,

$$f(x) = \frac{|x-2|}{x-2} = \begin{cases} \frac{x-2}{x-2} & \text{if } x > 2 \\ -\frac{(x-2)}{x-2} & \text{if } x < 2 \end{cases}$$
$$= \begin{cases} 1 & \text{if } x > 2 \\ -1 & \text{if } x < 2 \end{cases}$$

Therefore,

$$\lim_{x\to 2^-} \frac{|x-2|}{x-2} = -1 \text{ and } \lim_{x\to 2^+} \frac{|x-2|}{x-2} = 1.$$

Since,

$$\lim_{x\to 2^-} \frac{|x-2|}{x-2} \neq \lim_{x\to 2^+} \frac{|x-2|}{x-2}$$

the limit,

$$\lim_{x\to 2} \frac{|x-2|}{x-2}$$

does not exist. Let's consider a couple of more straight forward examples.

$$\lim_{x\to -1} \frac{x^2-5x+2}{2x^3+3x+1}$$
$$= \frac{\lim_{x\to -1}(x^2-5x+2)}{\lim_{x\to -1}(2x^3+3x+1)}$$
$$= \frac{1+5+2}{-2-3+1}$$
$$= \frac{8}{-4}$$
$$= -2$$

The different ways to evaluate indeterminate forms of limits include,

1. direct substitution
2. factoring
3. rationalizing
4. change of variable
5. one-sided limits

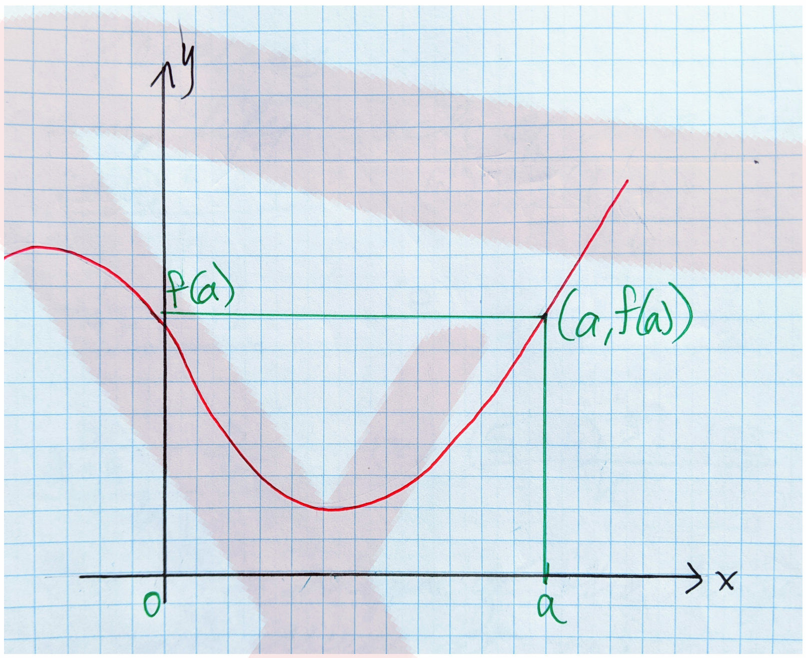
Now that we have a formal definition and understanidng of the limit of a function, we can define *continuity of a function at a point.*

Continuity Definition

The function $f(x)$ is continuous at $x = a$ if $f(a)$ is defined and if

$$\lim_{x \to a} f(x) = f(a)$$

Otherwise, $f(x)$ is discontinuous at $x = a$.



Exercises

1. Evaluate the limits of the indeterminant quotients.

a)
$$\lim_{x\to 2}\frac{4-x^2}{2-x}$$

b)
$$\lim_{x\to 0}\frac{7x-x^2}{x}$$

c)
$$\lim_{x\to -\frac{4}{3}}\frac{3x^2+x-4}{3x+4}$$

d)
$$\lim_{x\to 3}\frac{x^3-27}{x-3}$$

e)
$$\lim_{x\to -2}\frac{x^3+2x^2-4x-8}{x+2}$$

f)
$$\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}$$

g)
$$\lim_{x\to 0}\frac{2-\sqrt{4+x}}{x}$$

h)
$$\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}$$