Optimization Problems

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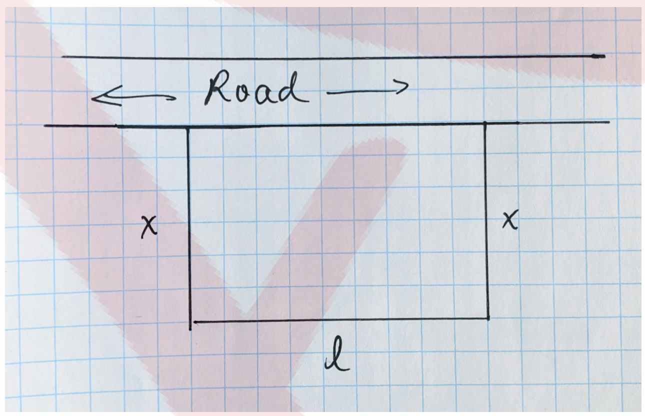
Optimization Problems

What is an optimization problem? What does optimization mean? The procedure of determining the maximum or minimum of a quantity is called *optimization* of that quantity. Let's consider an example to better understand optimization.

Example

A farmer has $800m$ of fencing and wants to enclose a rectangular filed. One side is closed off by a fence by a road. So only 3 sides of the filed need to be fenced in. What are the dimensions of the filed with greatest area?

Solutions: Let's draw the area to be fenced in with the information provided.



We have $800m$ of fencing so

$$800 = 2x + l \text{ or } 800 - 2x = l$$

The area of a rectangle is given by

$$A = xl = x(800 - 2x)$$

or

$$A(x) = 800x - 2x^2.$$

We know how to find the maximum of a function. Start by finding the point where the maximum or minimun may occur. We need to find the *criticial points* of our function $A(x) = 800x - 2x^2$. We do this by solving $A'(x) = 0$.

$$\begin{aligned} A'(x) &= 800 - 4x = 0 \\ \frac{800}{4} &= \frac{4x}{4} \\ 200 &= x \end{aligned}$$

The width of the field with the maximum area enclosed by $800m$ is $200m = x$. Now we need to find the length. We use the original relationship for the fencing, $800 = 2x + l$.

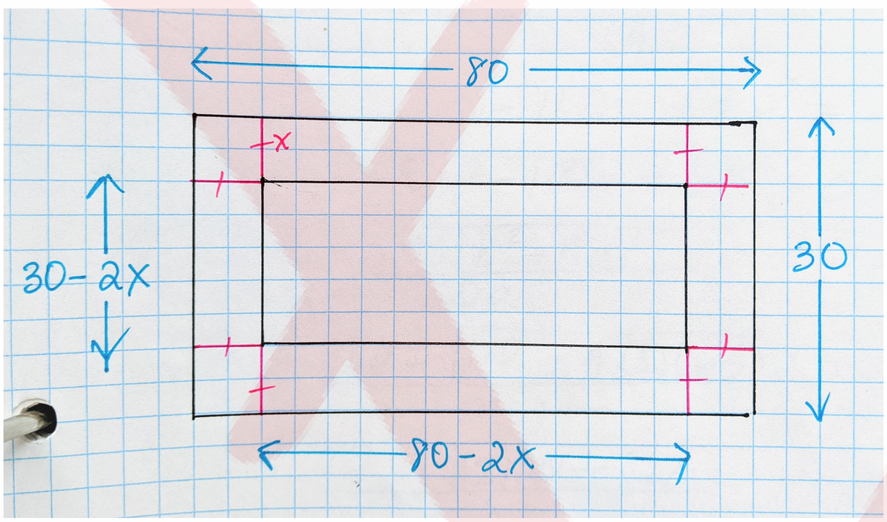
$$\begin{aligned} l &= 800 - 2x = 800 - 2(200) \\ &= 800 - 400 = 400m \end{aligned}$$

Therefore, the dimensions of the fenced area are $200m \times 400m$. Let's try another example.

Example

A piece of sheet metal, $80cm \times 30cm$ is to be used to make a rectangule box with an open top. Find the dimensions that will give the box the largest volume.

Solutions:



$$\begin{aligned} \text{volumne} &= lwh \\ &= (80 - 2x)(30 - 2x)x \\ &= (240 - 160x - 60x - 60x + 4x^2)x \\ &= (240 - 220x + 4x^2)x \\ V(x) &= 240x - 220x^2 + 4x^3 \end{aligned}$$

We need to find the critical points of the function for the volume $V(x)$.

$$\begin{aligned} V'(x) &= \frac{240 - 440x + 12x^2}{4} = \frac{0}{4} \\ 0 &= 60 - 110x + 3x^2 \\ 0 &= 3x^2 - 110x + 60 \end{aligned}$$

We need to use the quadratic formula to find the zeros of the quadratic. Recall the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

From here we have,

$$\begin{aligned} x &= \frac{110 \pm \sqrt{110^2 - 4(3)(60)}}{2(3)} \\ &= \frac{110 \pm \sqrt{12100 - 720}}{6} \\ &= \frac{110 \pm \sqrt{11380}}{6} \\ &= \frac{110 \pm 110668}{6} \\ \therefore x &= 36.11 \text{ or } 0.55 \end{aligned}$$

From here we see that $x = 36.11cm$ is not a reasonable height so$x = 0, 0.55cm$ are the possible heights. Therefore,

$$\begin{aligned} v''(x) &= -440 + 24(0.55) \\ &= -440 + 13.2 \\ &= -426.8 \end{aligned}$$

Therefore, the maximum volume occurs when $x = 0.55cm$ since $V'(0.55) < 0$. The maximum volume of the box is,

$$v(0.55) = 240(0.55) - 220(0.55)^2 + 4(0.55)^3 = 66.12cm^3$$

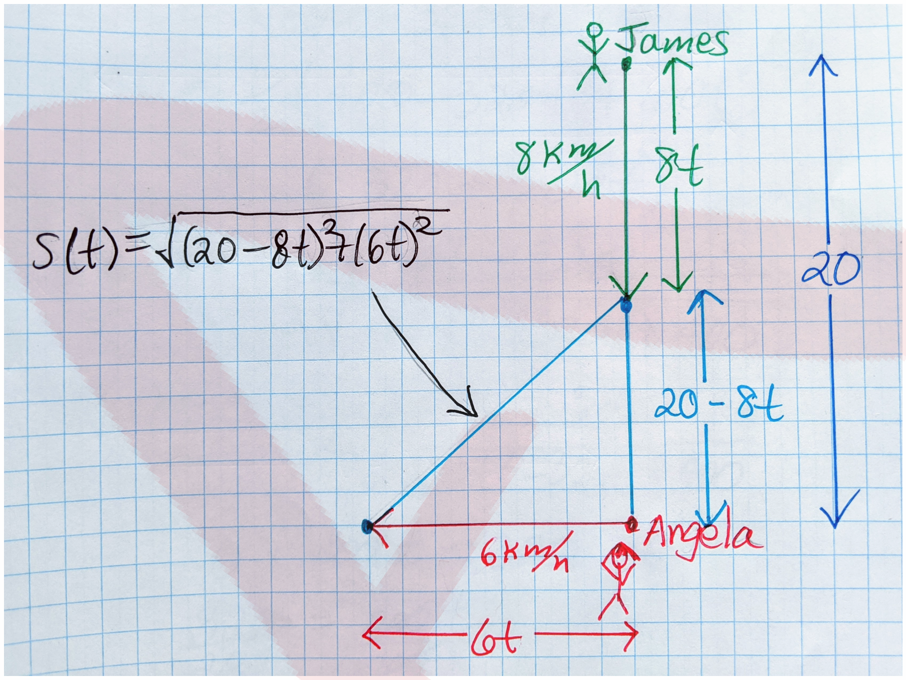
with dimensions,

$$80 - 2(0.55) = 78.9cm, \quad 30 - 2(0.55) = 78.9cm \text{ and } 0.55cm$$

Example

James and Angela both run each morning. James' house is 20km north of Angela's. Both start at 9am. James leaves his house and runs south at 8km/h. At the same time Angela leaves her house and runs west at 6km/h. When are James and Angela closest after 2.5h?

Solutions:



$$
\begin{aligned}
s(t) &= \sqrt{(20-8t)^2+(6t)^2} \\
&= \sqrt{400-320t+64t^2+36t^2} \\
&= \sqrt{400-320t+100t^2} \\
s'(t) &= \frac{1}{2}(400-320t+100t^2)^{-1/2}(-320+200t) \\
&= \frac{200t-320}{2\sqrt{100t^2-320t+400}} \\
s''(t) &= 0 = \frac{200t-320}{2\sqrt{100t^2-320t+400}} \\
0 &= 200t-320 \\
\frac{320}{200} &= \frac{200t}{200} \\
1.6 &= t
\end{aligned}
$$

The domain for t is $0 \leq t \leq 2.5$. We need to check which value of t, $t = 0, 1.6$ or 2.5 gives the smalles distance $s(t)$ value.

$$\begin{aligned}
s(0) &= \sqrt{(20 - 8(0))^2 + (6(0))^2} \\
&= \sqrt{(20)^2 + 0} \\
&= 20 \\
s(1.6) &= \sqrt{(20 - 8(1.6))^2 + (6(1.6))^2} \\
&= \sqrt{51.84 + 92.16} \\
&= \sqrt{144} \\
&= 12 \\
s(2.5) &= \sqrt{(20 - 8(2.5))^2 + (6(2.5))^2} \\
&= \sqrt{0 + 15^2} \\
&= 15
\end{aligned}$$

At time $t = 1.6$ hours, James and Angela are at their closest after 1.6 hours of their 2.5 hour run. So at time 10:36am Angela and James are at their closest.

Where can optimization be used?

There are problems in all industries where some quantity needs to maximized or minimized. There are many problems in economics and science that require optimization. Let's consider an example.

Exercises