Optimization Problems

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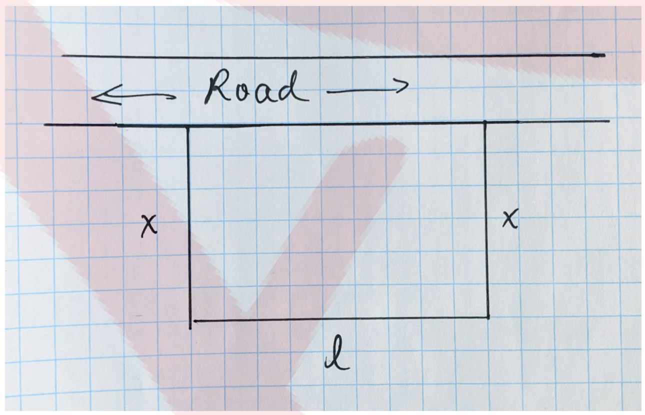
Optimization Problems

What is an optimization problem? What does optimization mean? The procedure of determining the maximum or minimum of a quantity is called *optimization* of that quantity. Let's consider an example to better understand optimization.

Example

A farmer has $800m$ of fencing and wants to enclose a rectangular filed. One side is closed off by a fence by a road. So only 3 sides of the filed need to be fenced in. What are the dimensions of the filed with greatest area?

Solutions: Let's draw the area to be fenced in with the information provided.



We have $800m$ of fencing so

$$800 = 2x + l \text{ or } 800 - 2x = l$$

The area of a rectangle is given by

$$A = xl = x(800 - 2x)$$

or

$$A(x) = 800x - 2x^2.$$

We know how to find the maximum of a function. Start by finding the point where the maximum or minimun may occur. We need to find the *criticial points* of our function $A(x) = 800x - 2x^2$. We do this by solving $A'(x) = 0$.

$$\begin{aligned} A'(x) &= 800 - 4x = 0 \\ \frac{800}{4} &= \frac{4x}{4} \\ 200 &= x \end{aligned}$$

The width of the field with the maximum area enclosed by $800m$ is $200m = x$. Now we need to find the length. We use the original relationship for the fencing, $800 = 2x + l$.

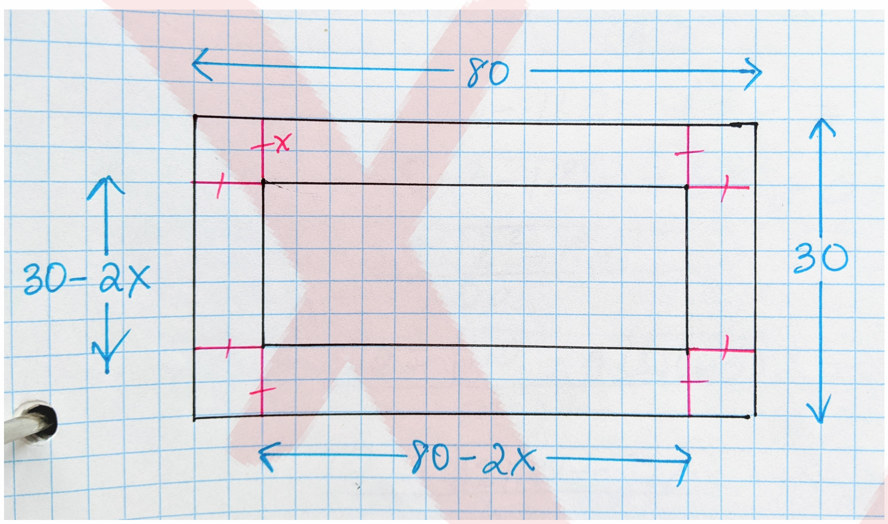
$$\begin{aligned} l &= 800 - 2x = 800 - 2(200) \\ &= 800 - 400 = 400m \end{aligned}$$

Therefore, the dimensions of the fenced area are $200m \times 400m$. Let's try another example.

Example

A piece of sheet metal, $80cm \times 30cm$ is to be used to make a rectangule box with an open top. Find the dimensions that will give the box the largest volume.

Solutions:



$$\begin{aligned} \text{volumne} &= lwh \\ &= (80 - 2x)(30 - 2x)x \\ &= (240 - 160x - 60x - 60x + 4x^2)x \\ &= (240 - 220x + 4x^2)x \\ V(x) &= 240x - 220x^2 + 4x^3 \end{aligned}$$

We need to find the critical points of the function for the volume $V(x)$.

$$\begin{aligned} V'(x) &= \frac{240 - 440x + 12x^2}{4} = \frac{0}{4} \\ 0 &= 60 - 110x + 3x^2 \\ 0 &= 3x^2 - 110x + 60 \end{aligned}$$

We need to use the quadratic formula to find the zeros of the quadratic. Recall the quadratic formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

From here we have,

$$\begin{aligned} x &= \frac{110 \pm \sqrt{110^2 - 4(3)(60)}}{2(3)} \\ &= \frac{110 \pm \sqrt{12100 - 720}}{6} \\ &= \frac{110 \pm \sqrt{11380}}{6} \\ &= \frac{110 \pm 110668}{6} \\ \therefore x &= 36.11 \text{ or } 0.55 \end{aligned}$$

From here we see that $x = 36.11cm$ is not a reasonable height so$x = 0, 0.55cm$ are the possible heights. Therefore,

$$\begin{aligned} v''(x) &= -440 + 24(0.55) \\ &= -440 + 13.2 \\ &= -426.8 \end{aligned}$$

Therefore, the maximum volume occurs when $x = 0.55cm$ since $V'(0.55) < 0$. The maximum volume of the box is,

$$v(0.55) = 240(0.55) - 220(0.55)^2 + 4(0.55)^3 = 66.12cm^3$$

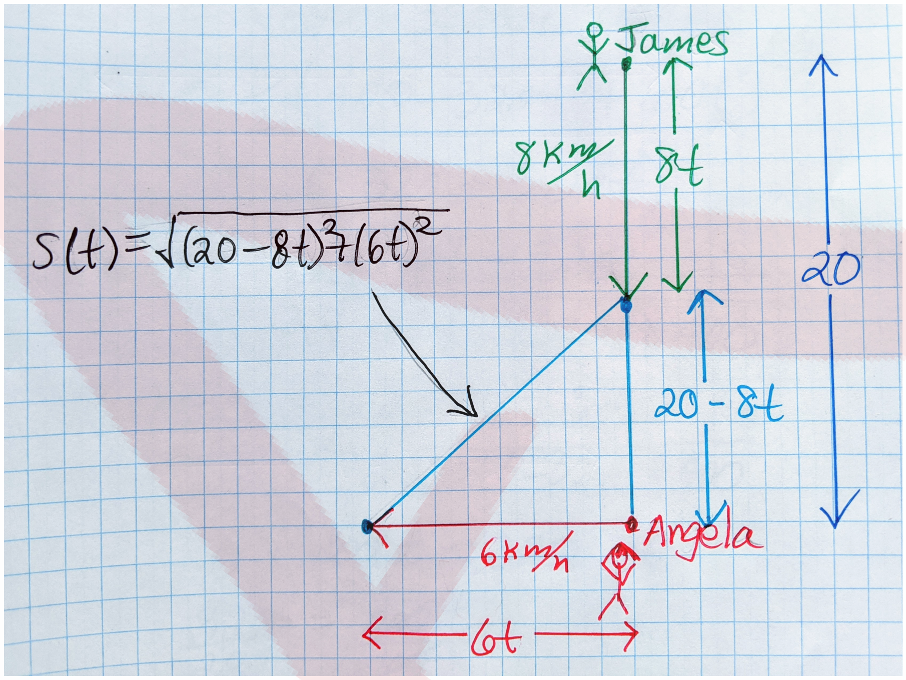
with dimensions,

$$80 - 2(0.55) = 78.9cm, \quad 30 - 2(0.55) = 78.9cm \text{ and } 0.55cm$$

Example

James and Angela both run each morning. James' house is 20km north of Angela's. Both start at 9am. James leaves his house and runs south at 8km/h. At the same time Angela leaves her house and runs west at 6km/h. When are James and Angela closest after 2.5h?

Solutions:



$$
\begin{aligned}
s(t) &= \sqrt{(20-8t)^2+(6t)^2} \\
&= \sqrt{400-320t+64t^2+36t^2} \\
&= \sqrt{400-320t+100t^2} \\
s'(t) &= \frac{1}{2}(400-320t+100t^2)^{-1/2}(-320+200t) \\
&= \frac{200t-320}{2\sqrt{100t^2-320t+400}} \\
s''(t) &= 0 = \frac{200t-320}{2\sqrt{100t^2-320t+400}} \\
0 &= 200t-320 \\
\frac{320}{200} &= \frac{200t}{200} \\
1.6 &= t
\end{aligned}
$$

The domain for t is $0 \leq t \leq 2.5$. We need to check which value of t, $t = 0, 1.6$ or 2.5 gives the smalles distance $s(t)$ value.

$$\begin{aligned}
s(0) &= \sqrt{(20-8(0))^2+(6(0))^2} \\
&= \sqrt{(20)^2+0} \\
&= 20 \\
s(1.6) &= \sqrt{(20-8(1.6))^2+(6(1.6))^2} \\
&= \sqrt{51.84+92.16} \\
&= \sqrt{144} \\
&= 12 \\
s(2.5) &= \sqrt{(20-8(2.5))^2+(6(2.5))^2} \\
&= \sqrt{0+15^2} \\
&= 15
\end{aligned}$$

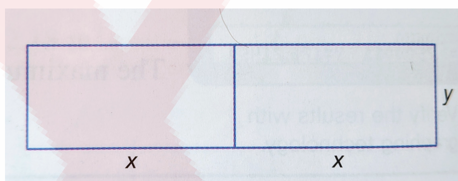
At time $t = 1.6$ hours, James and Angela are at their closest after 1.6 hours of their 2.5 hour run. So at time 10:36am Angela and James are at their closest.

Where can optimization be used?

There are problems in all industries where some quantity needs to maximized or minimized. There are many problems in economics and science that require optimization. Let's consider an example.

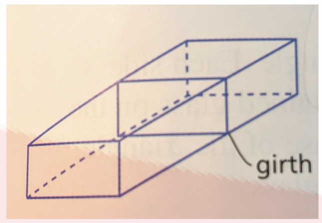
Exercises

1. Find two positive real numbers whose sum is 70 and whose product is as large as possible.
2. Find two positive real numbers such that the sum of the nubmers is 100. The product of the first number and two times the second number should be as a large as possible.
3. Find two integers whose difference is 22 and whose product is as small as possible.
4. Find two numbers whose sum is 190 such that the sum of the squares of the two numbers is minimized.
5. A rectangle has a perimeter of 440cm. What dimensions maximize the rectangle's area?
6. A rancher has 100m of fencing to enclose two rectangular corrals. The corrals have the same dimensions and one side is common. What dimensions will maximize the enclosed area?



7. A farmer has 60m of fencing to enclose a rectangular pen beside a large barn. The farmer uses the wall of the barn as one side of the fence. Find the maximum are that can be enclosed.
8. Tom makes an open box from a rectangular piece of metal by cutting equalt squares from the four corners and turning up the sides The piece of metal measures 60cm by 100cm. What are the dimensions of the box with the maximum volume?
9. Sandy will make a closed rectangular jewellery box with a square base from two different woods. The wood for the top and bottom costs $\$20/m^2$. The wood for the sides costs $\$30/m^2$. Find the dimensions that minimizes the wood costs for a volume of $4000cm^3$.
10. What is the maximum volume of a right cylinder that has a total surface area of $384\pi cm^2$?

11. Suppose that a courier accepts packages only if the sum of the length and the girth is no more than 300cm. The girth is the perimeter of the cross section of the package. What is the maximum volume of an acceptable rectangular box with a square cross section?



12. The landlord of a 40 unit apartment building is planning to increase the rent. Currently residents pay $\$700/month$. Four units are vacant. A real estate agency has found that, in this market, every \$25 increase in monthly rent results in one more vacant unit. What rent should the landlord charge to maximize revenue?

13. This Norman window is made up of a semicircle and a rectangle. The total perimeter of the window is 16m. What is the maximum area?