

Logarithmic Differentiation

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2020

Logarithmic Differentiation

Let's consider an example.

Example

For $y = x^x$, $x > 0$. Determine $\frac{dy}{dx}$.

Solution:

$$y = x^x, \text{ Take the natural logarithm, ln, of both sides.}$$

$$\ln y = \ln x^x$$

$$\ln y = x \ln x, \text{ Differentiate w.r.t. } x \text{ on both sides.}$$

$$\frac{1}{y} \frac{dy}{dx} = \ln x + \frac{x}{x}$$

$$\frac{1}{y} \frac{dy}{dx} = \ln x + 1$$

$$\frac{dy}{dx} = y(\ln x + 1)$$

$$\therefore \frac{dy}{dx} = x^x(\ln x + 1)$$

Example

Differentiate $y = x^n$.

Solution:

$$y = x^n$$

$$\ln y = \ln x^n, \text{ Take the natural logarithm, ln, of both sides.}$$

$$\ln y = n \ln x$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{n}{x}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{ny}{x} \\ &= \frac{nx^n}{x} \end{aligned}$$

$$\therefore \frac{dy}{dx} = nx^{n-1}$$

Example

Differentiate $y = (x^2 + 3)^x$.

Solution:

$$\begin{aligned}
 y &= (x^2 + 3)^x \\
 \ln y &= \ln(x^2 + 3)^x \\
 \ln y &= x \ln(x^2 + 3) \\
 \frac{1}{y} \frac{dy}{dx} &= \ln(x^2 + 3) = \frac{x(2x)}{x^2 + 3} \\
 &= \ln(x^2 + 3) + \frac{2x^2}{x^2 + 3} \\
 \frac{dy}{dx} &= y \left(\ln(x^2 + 3) + \frac{2x^2}{x^2 + 3} \right) \\
 \frac{dy}{dx} &= (x^2 + 3)^x \left(\ln(x^2 + 3) + \frac{2x^2}{x^2 + 3} \right)
 \end{aligned}$$

Example

Differentiate

$$y = \frac{(x^4 + 1)\sqrt{x + 2}}{(2x^2 + 2x + 1)}$$

Solution: We'll use logarithmic differentiation.

$$\begin{aligned}
 y &= \frac{(x^4 + 1)\sqrt{x + 2}}{(2x^2 + 2x + 1)} \\
 \ln y &= \ln \left(\frac{(x^4 + 1)\sqrt{x + 2}}{(2x^2 + 2x + 1)} \right) \\
 \ln y &= \ln(x^4 + 1) + \ln \sqrt{x + 2} - \ln(2x^2 + 2x + 1) \\
 \frac{1}{y} \frac{dy}{dx} &= \frac{4x^3}{x^4 + 1} + \frac{1}{2} \frac{1}{x + 2} - \frac{4x + 2}{2x^2 + 2x + 1} \\
 \frac{dy}{dx} &= y \left(\frac{4x^3}{x^4 + 1} + \frac{1}{2} \frac{1}{x + 2} - \frac{4x + 2}{2x^2 + 2x + 1} \right) \\
 &= \frac{(x^4 + 1)\sqrt{x + 2}}{(2x^2 + 2x + 1)} \left(\frac{4x^3}{x^4 + 1} + \frac{1}{2} \frac{1}{x + 2} - \frac{4x + 2}{2x^2 + 2x + 1} \right)
 \end{aligned}$$

Exercises

Differentiate.

a) $y = \ln(5x + 8)$

g) $v = e^t \ln t$

b) $s = \ln(t^3 - 2t^2 + 5)$

h) $y = (\ln x)^3$

c) $w = \ln \sqrt{z^2 + 3z}$

i) $x = \frac{e^t}{\ln t}$

d) $y = \ln \sqrt{x + 1}$

j) $f(x) = \ln \left(\frac{x^2 + 1}{x - 1} \right)$

e) $f(x) = x \ln x$

k) $g(z) = \ln(e^{-z} + ze^{-z})$

f) $y = \frac{\ln x}{x^2}$

l) $h(u) = e^{\sqrt{u}} \ln \sqrt{u}$

n) $f(x) = \ln(x^2 + 1)$

m) $g(x) = e^{2x-1} \ln(2x - 1)$

o) $f(x) = (x^2 + 1)^{-1} \ln(x^2 + 1)$