Factors 3

Raise My Marks

RaiseMyMarks.com

2020

What is a factor of a function?

Specifically, a polynomial function? A **linear factor** $x - a$ of a function $f(x)$ is such that $f(a) = 0$. For example, show that $x - 2$ is a factor of $x^3 - 3x^2 + 5x - 6$. Let $a = 2$. Then we have,

$$\begin{aligned} f(x) &= x^3 - 3x^2 + 5x - 6 \\ f(2) &= 8 - 12 + 10 - 6 \\ &= -4 + 4 \\ &= 0 \end{aligned}$$

or $x - 2$ is a factor of $f(x)$ when the remainder of $f(x) \div (x - 2)$ is 0. Let's check this is true.

$$\begin{array}{r|l} & x^2 - x + 3 \\ \hline x - 2 & x^3 - 3x^2 + 5x - 6 \\ & -(x^3 - 2x^2) \\ & \quad -x^2 + 5x \\ & \quad -(-x^2 + 2x) \\ & \qquad 3x - 6 \\ & \qquad -(3x - 6) \\ & \qquad\qquad 0 \end{array}$$

So the remainder $r = 0$. Therefore, $x - 2$ is a facor of $x^3 - 3x^2$. This is summarized nicely in the **Factor Theorem**,

Factor Theorem

$x - p$ is a factor of $f(x)$ if and only if $f(p) = 0$.

Some special polynomials and factors:

The sum and difference of cubes:

$$\begin{aligned} x^3 - a^3 &= (x - a)(x^2 + xa + a^2) \\ x^3 + a^3 &= (x + a)(x^2 - xa + a^2) \end{aligned}$$

Difference of squares

$$x^2 - a^2 = (x - a)(x + a)$$

Quadratic factors

$$ax^2 + bx + c \;=\; \left(x - \frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(x - \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$

where the quadratic formula has been used to find the roots and hence the factors of the quadratic. In general a polynomial of degree n has at most n real roots. It

is possible for a polynomial of degree n to have less than n real roots but every polynomial of degree nn has exactly n roots, real or complex.

Some properties of Quadratics

There are some interesting properties of quadratics. Let's consider the quadratic

$$q(x) = ax^2 + bx + c,$$

where a, b, c are real numbers.

1. The sum of the roots of $q(x) = 0 = ax^2 + bx + c$ is $-\frac{b}{a}$.
2. The products of the roots of $q(x) = 0 = ax^2 + bx + c$ is $\frac{c}{a}$.
3. Any quadratic equation can be written as,

$$\begin{aligned}
0 &= x^2 - (\text{ sum of roots })x + (\text{ product of roots }) \\
0 &= ax^2 + bx + c \\
0 &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \\
0 &= a\left(x^2 - \left(-\frac{b}{a}\right)x + \frac{c}{a}\right)
\end{aligned}$$

Note: Factors of functions are not necessarily just linear. The following function has quadratic factors,

$$f(x) = (x^2 + 1)^2$$

You may think of a factor of a function $f(x)$ as any functions $g_1(x), \ldots, g_n(x)$ such that $f(x) = g_1(x) \cdots g_n(x)$, where at least two of the functiosn $g_i(x)$ and $g_j(x)$ for $i \neq j$ are NOT equal to 1. Mos the functions we will consider will have factors that are linear or polynomial.

Exercises

1. Factor using the sum or difference of cubes formula.

   a) $x^3 - 27$

   b) $y^3 + 8$

   c) $2000w^3 + 2y^3$

   d) $125u^3 - 64r^3$

   e) $(x + y)^3 - u^3z^3$

2. If $(x-3)$ is a factor of $x^3 - 2x^2 - kx - 3$, what is k?

3. If $x^3 + 4x^2 + kx5$ is divisible by $x + 2$, what is k?

4. Show $x - y$ is a factor of $x^5 - y^5$.