Factors 2

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What is a factor of a function?

Specifically, a polynomial function? A **linear factor** $x-a$ of a function $f(x)$ is such that $f(a)=0$. For example, show that $x-2$ is a factor of x^3-3x^2+5x-6. Let $a=2$. Then we have,

$$
\begin{aligned}
f(x) &= x^3-3x^2+5x-6 \\
f(2) &= 8-12+10-6 \\
&= -4+4 \\
&= 0
\end{aligned}
$$

or $x-2$ is a factor of $f(x)$ when the remainder of $f(x)\div(x-2)$ is 0. Let's check this is true.

$$
\begin{array}{r|l}
 & x^2-\;x\;+\;3 \\
\hline
x-2 & x^3-3x^2+5x-6 \\
 & -(x^3-2x^2) \\
 & \quad -x^2+5x \\
 & \quad -(-x^2+2x) \\
 & \qquad 3x-6 \\
 & \qquad -(3x-6) \\
 & \qquad\qquad 0
\end{array}
$$

So the remainder $r=0$. Therefore, $x-2$ is a facor of x^3-3x^2. This is summarized nicely in the **Factor Theorem**,

Factor Theorem

$$x-p \text{ is a factor of } f(x) \text{ if and only if } f(p)=0.$$

Some special polynomials and factors:

The sum and difference of cubes:

$$
\begin{aligned}
x^3-a^3 &= (x-a)(x^2+xa+a^2) \\
x^3+a^3 &= (x+a)(x^2-xa+a^2)
\end{aligned}
$$

Difference of squares

$$x^2 - a^2 = (x-a)(x+a)$$

Quadratic factors

$$ax^2 + bx + c = \left(x - \frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(x - \frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$

where the quadratic formula has been used to find the roots and hence the factors of the quadratic. In general a polynomial of degree n has at most n real roots. It

is possible for a polynomial of degree n to have less than n real roots but every polynomial of degree nn has exactly n roots, real or complex.

Some properties of Quadratics

There are some interesting properties of quadratics. Let's consider the quadratic

$$q(x) = ax^2 + bx + c,$$

where a, b, c are real numbers.

1. The sum of the roots of $q(x) = 0 = ax^2 + bx + c$ is $-\frac{b}{a}$.
2. The products of the roots of $q(x) = 0 = ax^2 + bx + c$ is $\frac{c}{a}$.
3. Any quadratic equation can be written as,

$$\begin{aligned} 0 &= x^2 - (\text{ sum of roots })x + (\text{ product of roots }) \\ 0 &= ax^2 + bx + c \\ 0 &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \\ 0 &= a\left(x^2 - \left(-\frac{b}{a}\right)x + \frac{c}{a}\right) \end{aligned}$$

Note: Factors of functions are not necessarily just linear. The following function has quadratic factors,

$$f(x) = (x^2 + 1)^2$$

You may think of a factor of a function$f(x)$ as any functions $g_1(x), \ldots, g_n(x)$ such that $f(x) = g_1(x) \cdots g_n(x)$, where at least two of the functiosn $g_i(x)$ and $g_j(x)$ for $i \neq j$ are NOT equal to 1. Mos the functions we will consider will have factors that are linear or polynomial.

Exercises

1. Factor the following fucntions completely.

   a) $x^4 - 6x^3 - 15x^2 - 6x - 16$

   b) $x^3 + 2x^2 - x - 2$

   c) $y^3 + 19y^2 - 19y - 1$

   d) $x^3 - 4x + 3$

   e) $x^4 - 8x^3 + 3x^2 + 40x - 12$

2. If $(x+2)$ is a factor of $x^3 - \frac{6}{k}x^2 - 5x + 6$, what is k?

3. Show $x - y$ is a factor of $x^n - y^n$.