Factors

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What is a factor of a function?

Specifically, a polynomial function? A **linear factor** $x-a$ of a function $f(x)$ is such that $f(a)=0$. For example, show that $x-2$ is a factor of x^3-3x^2+5x-6. Let $a=2$. Then we have,

$$\begin{aligned} f(x) &= x^3-3x^2+5x-6 \\ f(2) &= 8-12+10-6 \\ &= -4+4 \\ &= 0 \end{aligned}$$

or $x-2$ is a factor of $f(x)$ when the remainder of $f(x) \div (x-2)$ is 0. Let's check this is true.

$$\begin{array}{rl} & x^2 - x + 3 \\ x-2 \,\big) & \overline{x^3 - 3x^2 + 5x - 6} \\ & \underline{-(x^3 - 2x^2)} \\ & \quad -x^2 + 5x \\ & \quad \underline{-(-x^2 + 2x)} \\ & \qquad 3x - 6 \\ & \qquad \underline{-(3x - 6)} \\ & \qquad\qquad 0 \end{array}$$

So the remainder $r=0$. Therefore, $x-2$ is a facor of x^3-3x^2. This is summarized nicely in the **Factor Theorem**,

Factor Theorem

$x-p$ is a factor of $f(x)$ if and only if $f(p)=0$.

Some special polynomials and factors:

The sum and difference of cubes:

$$\begin{aligned} x^3-a^3 &= (x-a)(x^2+xa+a^2) \\ x^3+a^3 &= (x+a)(x^2-xa+a^2) \end{aligned}$$

Difference of squares

$$x^2 - a^2 = (x-a)(x+a)$$

Quadratic factors

$$ax^2 + bx + c = \left(x - \frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(x - \frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$

where the quadratic formula has been used to find the roots and hence the factors of the quadratic. In general a polynomial of degree n has at most n real roots. It

is possible for a polynomial of degree n to have less than n real roots but every polynomial of degree nn has exactly n roots, real or complex.

Some properties of Quadratics

There are some interesting properties of quadratics. Let's consider the quadratic

$$q(x) = ax^2 + bx + c,$$

where a, b, c are real numbers.

1. The sum of the roots of $q(x) = 0 = ax^2 + bx + c$ is $-\frac{b}{a}$.
2. The products of the roots of $q(x) = 0 = ax^2 + bx + c$ is $\frac{c}{a}$.
3. Any quadratic equation can be written as,

$$\begin{aligned}
0 &= x^2 - (\text{ sum of roots })x + (\text{ product of roots }) \\
0 &= ax^2 + bx + c \\
0 &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \\
0 &= a\left(x^2 - \left(-\frac{b}{a}\right)x + \frac{c}{a}\right)
\end{aligned}$$

Note: Factors of functions are not necessarily just linear. The following function has quadratic factors,

$$f(x) = (x^2 + 1)^2$$

You may think of a factor of a function$f(x)$ as any functions $g_1(x), \ldots, g_n(x)$ such that $f(x) = g_1(x) \cdots g_n(x)$, where at least two of the functiosn $g_i(x)$ and $g_j(x)$ for $i \neq j$ are NOT equal to 1. Mos the functions we will consider will have factors that are linear or polynomial.

Exercises

1. For the given linear function and function $f(x)$, is the linear function and factor of $f(x)$? Explain why or why not.

   a) $x - 1,\ f(x) = x^2 - 7x + 6$

   b) $x - 3,\ f(x) = x^3 + 6x^2 - 2x + 3$

   c) $x + 1,\ f(x) = x^3 - 5x^2 - 4x + 3$

   d) $2x - 1,\ f(x) = 4x^3 - 6x^2 + 8x - 3$

e) $x - 2,\ f(x) = x^3 - 3x^2 - 4x + 12$

2. If $(x - 1)$ is a factor of $x^3 - 2kx^2 + 3x + 1$, what is k?

3. Show $x - y$ is a factor of $x^4 - y^4$.