Asymptotes 4

RaiseMyMarks.com

2020

Vertical Asymptotes

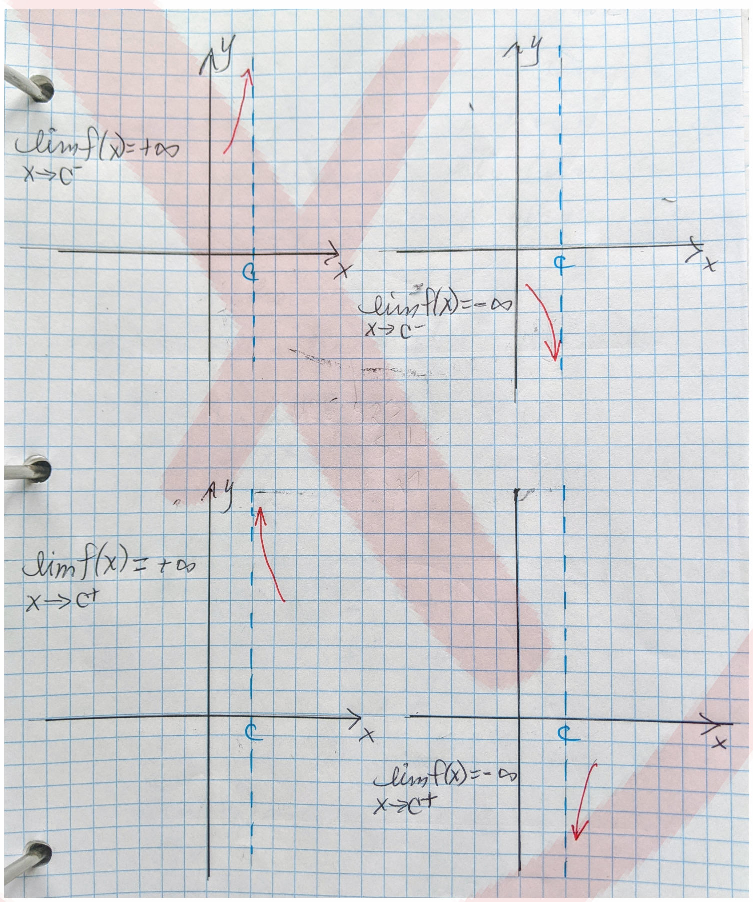
Vertical asymptotes at $x = c$ occurs when one of the following limits occur:

$$\lim_{x\to c^+} f(x) \text{ or } \lim_{x\to c^-} f(x)$$

does not exist. That is,

$$\lim_{x\to c^+} f(x) = +\infty$$
$$\lim_{x\to c^+} f(x) = -\infty$$
$$\lim_{x\to c^-} f(x) = +\infty$$
$$\lim_{x\to c^-} f(x) = -\infty$$

For a general rational function $f(x) = \frac{p(x)}{q(x)}$, f has a vertical asymptote $x = c$ if $q(c) = 0$ and $p(c) \neq 0$.



Let's consider and example.

Example

Determine any vertical asymptotes of the function

$$f(x) = \frac{x}{x^2 + x - 2}$$

and describe the behavirous.

Solution:

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x) = x$ and $q(x) = x^2 + x - 2$.

$$q(x) = x^2 + x - 2 = (x + 2)(x - 1) \Rightarrow x = -2, 1$$

	-2	0	1	
x	-	-	+	+
$x + 2$	-	+	+	+
$x - 1$	-	-	-	+
$f(x)$	-	+	-	+

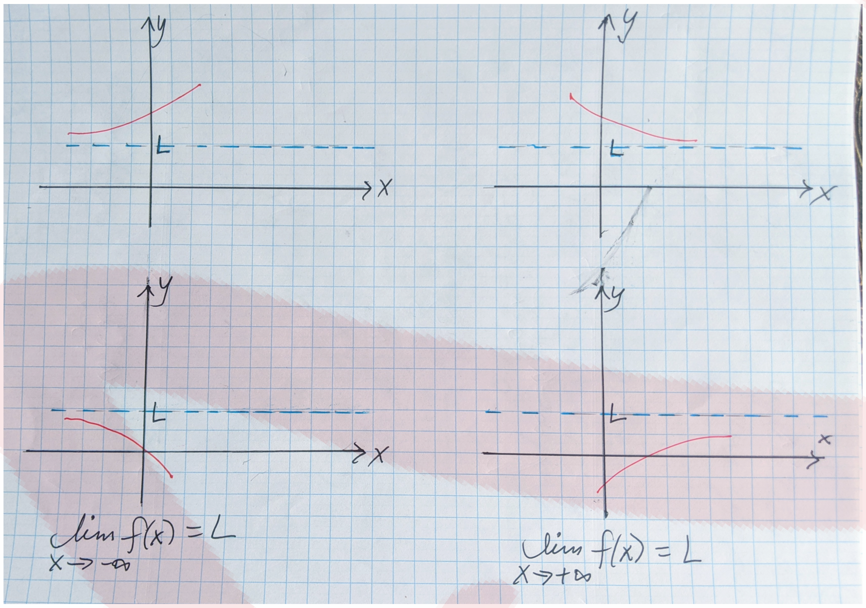
$$\lim_{x \to -2^-} f(x) = -\infty$$
$$\lim_{x \to -2^+} f(x) = +\infty$$
$$\lim_{x \to 1^-} f(x) = -\infty$$
$$\lim_{x \to x^+} f(x) = +\infty$$

Horizontal Asymptotes

When the limit of the function is a finite number as $x \to \infty$ or $x \to -\infty$. Or, if,

$$\lim_{x \to +\infty} f(x) = L \text{ or } \lim_{x \to -\infty} f(x) = L$$

then, we say the line $y = L$ is a horizatonal asymptote of $f(x)$.



Example

Determine the equation of all asymptotes of the graph

$$f(x) = \frac{2x^2 + 3x - 1}{x + 1}$$

Solution: To find the vertical asymptotes, we need to consider when the function $f(x)$ is undefined. $p(x) = 2x^2 + 3x - 1$ and $q(x) = x + 1 = 0$. Therefore, $x = -1$ is a vertical asymptote. Notes that $p(-1) = 2 - 3 - 1 = -2 \neq 0$. To find the horizontal asymptotes,

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{2x^2 + 3x - 1}{x + 1} \\
&= \lim_{x\to\infty} \frac{x^2\left(2 + \frac{3}{x} - \frac{1}{x^2}\right)}{x\left(1 + \frac{1}{x}\right)} \\
&= \lim_{x\to\infty} 2x = \infty, \quad \text{and} \\
\lim_{x\to-\infty} f(x) &= -\infty
\end{aligned}$$

Therefore, no horizontal asymptotes. In order to find any *oblique asymptotes*, we need to divide $q(x) = (x+1)$ into $p(x) = 2x^2 + 3x - 1$.

$$\begin{array}{r|l} & \quad 2x + \ 1 \\ \hline x+1 & \quad 2x^2 + 3x - 1 \\ & -(2x^2 + 2x) \\ \hline & \qquad\quad x - 1 \\ & \qquad -(x+1) \\ \hline & \qquad\qquad -2 \end{array}$$

Therefore,

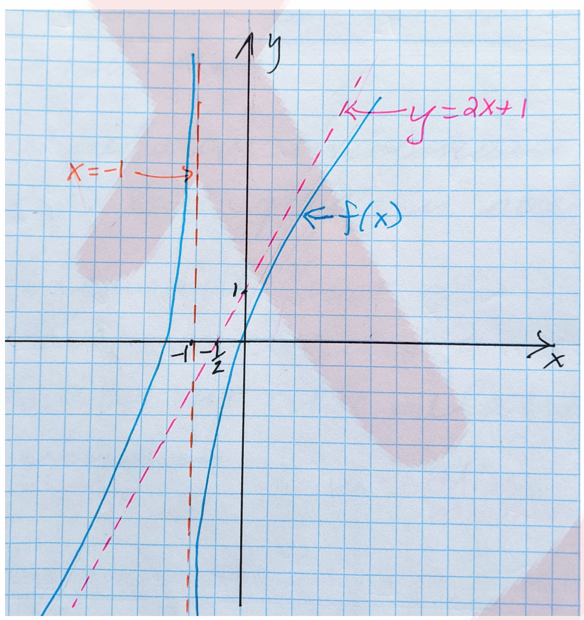
$$f(x) = \frac{2x^2 + 3x - 1}{x+1} = 2x + 1 - \frac{2}{x+1}$$

$$\lim_{x \to \infty} \frac{2}{x+1} = 0 \quad , \quad \lim_{x \to -\infty} \frac{2}{x+1} = 0$$

$$\therefore \text{ as } x \to \infty \quad , \quad f(x) \to 2x + 1 \text{ or}$$

$$\text{as } x \to -\infty \quad , \quad f(x) \to 2x + 1.$$

Therefore, $y = 2x + 1$ is an oblique asymptote.



Exercises

Evaluate $\lim_{x\to\infty} f(x)$ for the following functions,

a)

$$f(x) = \frac{2x+3}{x-1}$$

b)

$$f(x) = \frac{5x^2-3}{x^2+2}$$

c)

$$f(x) = \frac{-5x^2+3x}{2x^2-5}$$

d)

$$f(x) = \frac{2x^5-3x^2+5}{3x^4+5x-4}$$