Asymptotes

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Vertical Asymptotes

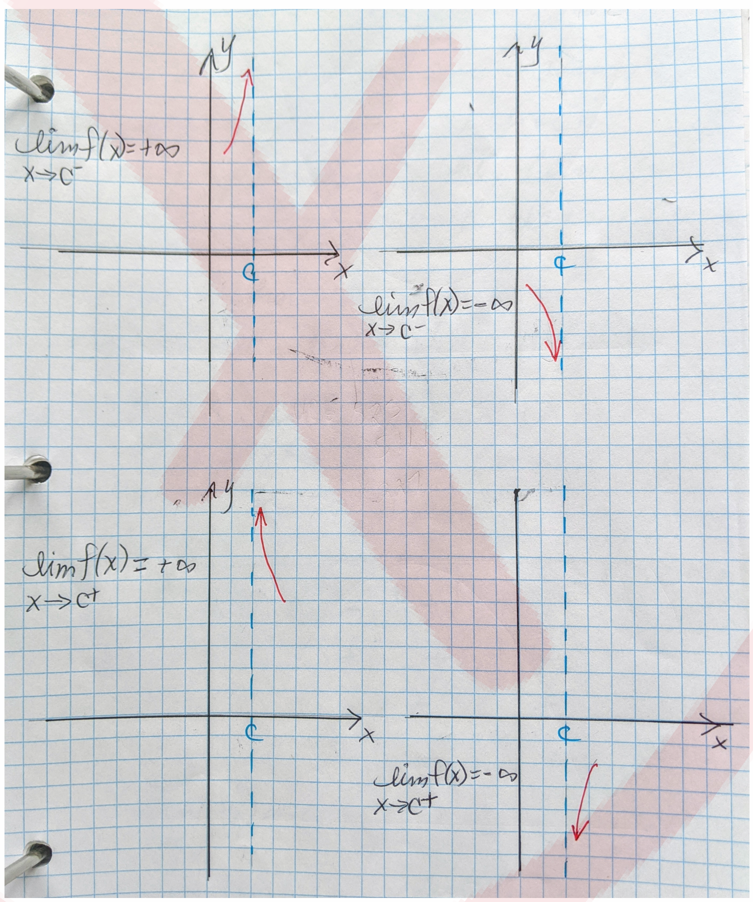
Vertical asymptotes at $x = c$ occurs when one of the following limits occur:

$$\lim_{x \to c^+} f(x) \text{ or } \lim_{x \to c^-} f(x)$$

does not exist. That is,

$$\lim_{x \to c^+} f(x) = +\infty$$
$$\lim_{x \to c^+} f(x) = -\infty$$
$$\lim_{x \to c^-} f(x) = +\infty$$
$$\lim_{x \to c^-} f(x) = -\infty$$

For a general rational function $f(x) = \frac{p(x)}{q(x)}$, f has a vertical asymptote $x = c$ if $q(c) = 0$ and $p(c) \neq 0$.



Let's consider and example.

Example

Determine any vertical asymptotes of the function

$$f(x) = \frac{x}{x^2 + x - 2}$$

and describe the behavirous.

Solution:

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x) = x$ and $q(x) = x^2 + x - 2$.

$$q(x) = x^2 + x - 2 = (x + 2)(x - 1) \Rightarrow x = -2, 1$$

	-2	0	1	
x	-	-	+	+
$x + 2$	-	+	+	+
$x - 1$	-	-	-	+
$f(x)$	-	+	-	+

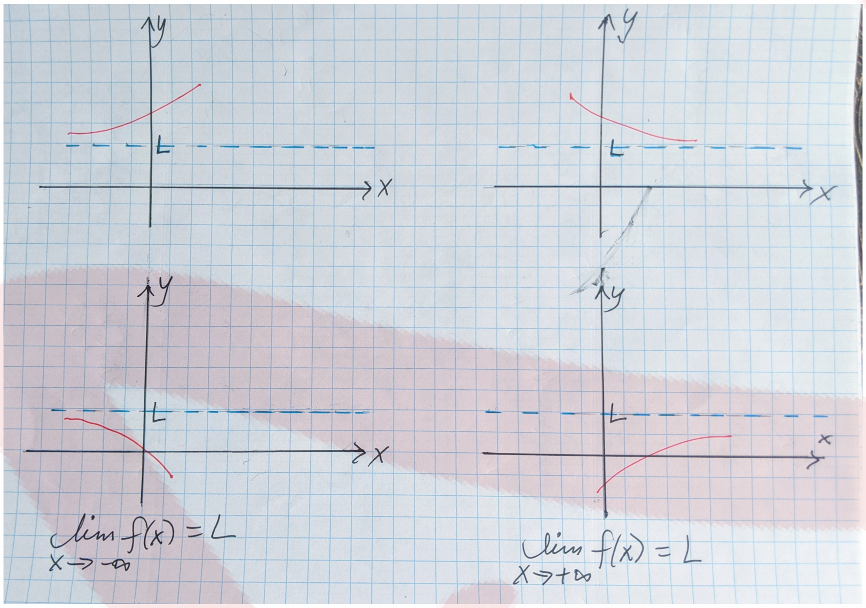
$$\lim_{x \to -2^-} f(x) = -\infty$$
$$\lim_{x \to -2^+} f(x) = +\infty$$
$$\lim_{x \to 1^-} f(x) = -\infty$$
$$\lim_{x \to x^+} f(x) = +\infty$$

Horizontal Asymptotes

When the limit of the function is a finite number as $x \to \infty$ or $x \to -\infty$. Or, if,

$$\lim_{x \to +\infty} f(x) = L \text{ or } \lim_{x \to -\infty} f(x) = L$$

then, we say the line $y = L$ is a horizatonal asymptote of $f(x)$.



Example

Determine the equation of all asymptotes of the graph

$$f(x) = \frac{2x^2+3x-1}{x+1}$$

Solution: To find the vertical asymptotes, we need to consider when the function $f(x)$ is undefined. $p(x) = 2x^2+3x-1$ and $q(x) = x+1 = 0$. Therefore, $x = -1$ is a vertical asymptote. Notes that $p(-1) = 2-3-1 = -2 \neq 0$. To find the horizontal asymptotes,

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{2x^2+3x-1}{x+1} \\
&= \lim_{x\to\infty} \frac{x^2\left(2+\frac{3}{x}-\frac{1}{x^2}\right)}{x\left(1+\frac{1}{x}\right)} \\
&= \lim_{x\to\infty} 2x = \infty, \quad \text{and} \\
\lim_{x\to-\infty} f(x) &= -\infty
\end{aligned}$$

Therefore, no horizontal asymptotes. In order to find any *oblique asymptotes*, we need to divide $q(x) = (x+1)$ into $p(x) = 2x^2 + 3x - 1$.

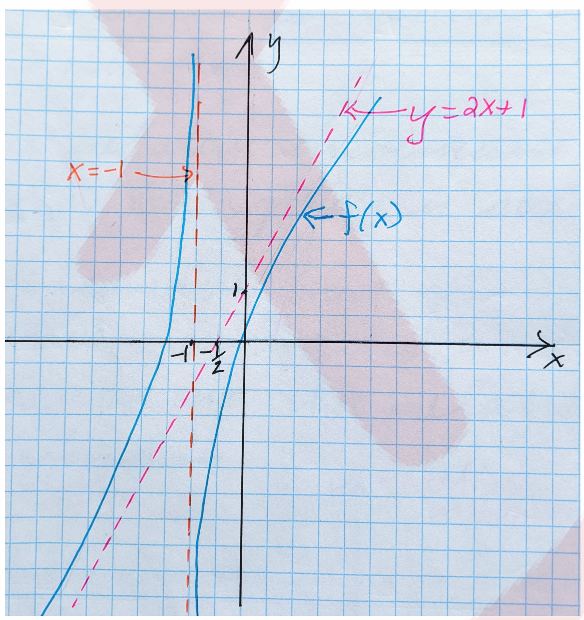
$$\begin{array}{r|l} & \quad 2x + 1 \\ \hline x+1 & 2x^2 + 3x - 1 \\ & \underline{-(2x^2 + 2x)} \\ & \qquad\quad x - 1 \\ & \qquad \underline{-(x+1)} \\ & \qquad\qquad -2 \end{array}$$

Therefore,

$$f(x) = \frac{2x^2 + 3x - 1}{x+1} = 2x + 1 - \frac{2}{x+1}$$

$$\lim_{x\to\infty} \frac{2}{x+1} = 0 \quad , \quad \lim_{x\to-\infty} \frac{2}{x+1} = 0$$
$$\therefore \text{ as } x \to \infty \quad , \quad f(x) \to 2x + 1 \text{ or}$$
$$\text{as } x \to -\infty \quad , \quad f(x) \to 2x + 1.$$

Therefore, $y = 2x + 1$ is an oblique asymptote.



Exercises

Determine the oblique asymptotes for the following functions.

a)

$$f(x) = \frac{3x^2 - 2x - 17}{x - 3}$$

b)

$$f(x) = \frac{2x^2 + 9x + 2}{2x + 3}$$

c)

$$f(x) = \frac{x^3 - 1}{x^2 + 2x}$$

d)

$$f(x) = \frac{x^3 - x^2 - 9x + 15}{x^2 - 4x + 3}$$

e)

$$f(x) = \frac{x^2 + 3x + 7}{x + 2}$$