Solving Trigonometric Equations 2

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Recall the *difference of squares*,

$$x^2 - a^2 \tag{1}$$

where a is any real number. Recall a *perfect square*,

$$x^2 + 2ax + a^2 \tag{2}$$

where a is any real number. Recall to factor each of these. The factorization for a difference of squares is given by,

$$x^2 - a^2 = (x - a)(x + a) \tag{3}$$

and the factorization for a perfect square is given by,

$$x^2 + 2ax + a^2 = (x + a)^2 \tag{4}$$

Let's have a quick look and recall of the **composition of functions**. Suppose we have two functions, $f(x) = x^2 - 1$ and $g(x) = \sin x$. The composition of the functions f and g is denoted by $f \circ g(x)$ and is evaluated as follows,

$$\begin{aligned} f \circ g(x) &= f(g(x)) \\ &= (\sin x)^2 - 1 \\ &= \sin^2 x - 1 \end{aligned}$$

Now our new function is $\sin^2 x - 1$. Let's consider an example.

Example

Solve the following equation,

$$\sin^2 x - 1 = 0 \tag{5}$$

Solution: When solving and equation we need to determine the values of x, the independent variable, that satisfy the equation (5) in this case. In this situation we can't really isolate our trigonometric function into the form $\sin\theta = b$ where b is any real number. Instead, we're going to have to factor the left hand side of (5) and see what we get. Notice that the left hand side of (5) is actually

$LHS = f \circ g(x) = f(g(x))$ where $f(x) = x^2 - 1$ and $g(x) = \sin x$ from above?

Can we factor $f(x) = x^2 - 1$? Yes. This is a difference of squares so from (3) we have,

$$f(x) = x^2 - 1 = (x-1)(x+1)$$

Now we can go ahead and factor the LHS of (5).

$$\begin{aligned} LHS = \sin^2 x - 1 &= f \circ g(x) \\ &= f(g(x)) \\ &= (g(x) - 1)(g(x) + 1) \\ &= (\sin x - 1)(\sin x + 1) \end{aligned}$$

Our equation in (5) can now be written as,

$$\sin^2 - 1 = (\sin x - 1)(\sin x + 1) = 0$$

When does the product equal zero? When at least one of the factors in the product equals zero. So in this case when,

$$\begin{aligned} & \sin x - 1 = 0 \text{ or } \sin x + 1 = 0 \\ \text{this is the same as,} \quad & \sin x = 1 \text{ or } \sin x = -1 \end{aligned}$$

What values of x satisfy either $\sin x = 1$ or $\sin x = -1$? When $x = 90°$ then $\sin 90° = 1$. When $x = 270°$ then $\sin 270° = -1$. Therefore, when $x = 90°$ or $270°$ is a solution to $\sin^2 x - 1 = 0$.

Exercises

Solve the following equations,

a) $2\sin^2 x = \sqrt{2}\sin x$

b) $\cot^2 x = -\cot x$

c) $\sqrt{3}\cot^2 x - 2\cot x - \sqrt{3} = 0$

d) $\sin^2 x - \cos^2 x + \sin x = 0$

e) $\sqrt{3}\cot^2 x - 2\cot x - \sqrt{3} = 0$

f) $\sin^2 x - \sin x = 0$

g) $2\cos^2 x = \sin^2 x - 1$

h) $2\cos^2 x - 7\cos x + 3 = 0$

i) $3\tan x - 1 = 2\tan x$

j) $\sqrt{2}\cos^2 x + 5\cos x + 2 = 0$

k) $\tan^2 x - (1 + \sqrt{3})\tan x + \sqrt{3} = 0$

l) $4\sin^3 x + 2\sin^2 x - 2\sin x - 1 = 0$