Rational Functions & Asymptotes

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Rational Functions

Let's consider functions of the form

$$f(x) = \frac{1}{x}$$

, or *rational functions*. These functions usually have *asymptotes*, vertical and/or horizontal. An *asymptote* is a line the functions approahces but never crosses or meets. Let's consider an examples.

Example

Find the asymptote(s) of the following function,

$$f(x) = \frac{1}{x}$$

Solution: To find the horizontal asymptote we need to consider the limit of the function as x approaches $+\infty$ or $-\infty$.

$$\begin{aligned} \lim_{x \longrightarrow +\infty} \frac{1}{x} &= 0 \text{ and} \\ \lim_{x \longrightarrow -\infty} \frac{1}{x} &= 0 \end{aligned}$$

Therefore, the vertical asymptote is given by the equation

$$y = 0$$

A vertical asymptote occurs when a function is undefined at a particular value. So in this case our function is undefined when $x = 0$.

$$\begin{aligned} \lim_{x \longrightarrow 0^-} \frac{1}{x} &= -\infty \text{ and} \\ \lim_{x \longrightarrow 0^+} \frac{1}{x} &= +\infty \end{aligned}$$

Now we can graph the function below,

Transformation of the parent function $f(x) = \frac{1}{x}$ will change the location of one or more of the asymptotes. For example,

Example

$$f(x) = \frac{1}{x-2} + 3$$

We have ahorizontal translation right 2 units; vertical translation up 3 units. This means that the asymptote locations of the parent function need to be transformed the same. So our vertical asymptote is now $x = 2$; the horizontal asymptote is now $y = 3$. We can verify this as follows:

$$\begin{aligned} \lim_{x \longrightarrow +\infty} \left(\frac{1}{x-2} + 3 \right) &= 3 \text{ and} \\ \lim_{x \longrightarrow -\infty} \left(\frac{1}{x-2} + 3 \right) &= 3 \end{aligned}$$

which gives a horizontal asymptote of $y = 3$; for the vertical asymptote,

$$\begin{aligned} \lim_{x \longrightarrow 2^+} \left(\frac{1}{x-2} + 3 \right) &= +\infty \text{ and} \\ \lim_{x \longrightarrow 2^-} \left(\frac{1}{x-2} + 3 \right) &= -\infty \end{aligned}$$

implying our vertical asymptote is $x = 2$. Let's graph the function but before we do, let's locate the y intercept and the *zeros* or x intercepts of the function. To find the y-intercept, let $x = 0$ and solve for y.

$$\begin{aligned} f(x) &= \frac{1}{x-2} + 3 \\ &= \frac{1}{-2} + 3 \\ \therefore y &= 2.5 \end{aligned}$$

Therefore, our y-intercept is $(0, 2.5)$. To find the x-intercept, let $y = 0$ and solve for

x.

$$
\begin{aligned}
f(x) &= \frac{1}{x-2} + 3 \\
0 &= \frac{1}{x-2} + 3 \\
-3 &= \frac{1}{x-2} \\
-3(x-2) &= 1 \\
-3x+6 &= 1 \\
-3x &= -5 \\
\therefore x &= \frac{5}{3}
\end{aligned}
$$

Therefore, our x-intercept is $(5/3, 0)$. With this information and the location of the asymptotes we can now graph the function $f(x) = \frac{1}{x-2} + 3$.

Exercises

1. Find all the asymptotes, horizontal and vertical, for the following functions,

   a) $f(x) = 1 + \frac{2}{x+5}$

   b) $f(x) = -6 - \frac{9}{x+2}$

   c) $f(x) = -\frac{3}{x+6}$

   d) $f(x) = -3 + \frac{1}{x-2}$

   e) $f(x) = \frac{4}{x-3} + 5$

2. Find the domain of the following functions,

   a) $f(x) = \frac{7}{x+7} + 5$

   b) $f(x) = \frac{4}{x-3} + 3$

   c) $f(x) = \frac{2}{x-3} + 8$

   d) $f(x) = -\frac{1}{x+1} + 5$

   e) $f(x) = \frac{3}{x+3} - 3$