L'Hôpital's Rule
Indeterminate forms of Limits

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Indeterminate forms of Limits

It is possible when evaluating a limit we run into some problems. For example, when we go through the steps of evaluating the limit and we end up in one of the following situations,

$$\frac{\infty}{\infty},\ \frac{0}{0},\ (0)(\pm\infty),\ 1^{\infty},\ 0^{0},\ \infty^{0},\ \infty-\infty \tag{1}$$

Any of these forms in (1) is referred to as an *indeterminate form* of a limit. When faced with any of these indeterminate forms the following "rule" is performed to hopefully eliminate the indeterminate form. This rule is called **l'Hôpital's Rule** given below,

l'Hôpital's Rule

Suppose,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0} \text{ or } \lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\pm\infty}{\pm\infty}$$

where a is any real number ∞ or $+\infty$. Then,

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Example

Find the following limit

$$\lim_{x\to 0}\frac{\sin x}{x}$$

Solution:

$$\begin{aligned}
&\lim_{x\to 0}\frac{\sin x}{x}\\
=\ &\frac{0}{0},\ \text{indeterminatne form so apply l'Hôpital's rule}\\
=\ &\lim_{x\to 0}\frac{\cos x}{1}\\
=\ &1
\end{aligned}$$

Example

Evaluate the following limit,

$$\lim_{x\to 0^+} x\ln x$$

Solution:

$$\begin{aligned} & \lim_{x \to 0^+} x \ln x \\ = & \; 0 \cdot \infty \end{aligned}$$

While $0 \cdot \infty$ is an indeterminate form it is not in one of the forms that is found in l'Hôpital's rule. So we need to try and rearrange the function so that we obtain either $\frac{0}{0}$ or $\frac{\infty}{\infty}$ as the indeterminate form.

$$\begin{aligned} & \lim_{x \to 0^+} x \ln x \\ = & \lim_{x \to 0^+} \frac{\ln x}{1/x} \text{ now we will evaluate the limit of this form of the function ad see what we get.} \\ = & \lim_{x \to 0^+} \frac{1/x}{-1/x^2} \\ = & \frac{\infty}{\infty} \text{ Now we can apply l'Hôpital's Rule} \\ = & \lim_{x \to 0^+} \frac{1}{x} \left(-\frac{x^2}{1} \right) \\ = & \lim_{x \to 0^+} (-x) \\ = & \; 0 \end{aligned}$$

Note:

$$f(x)g(x) = \frac{g(x)}{1/f(x)} = \frac{f(x)}{1/g(x)}$$

Example

Evaluate the following limit,

$$\lim_{x \to a} \frac{x - a}{x^2 - a^2}$$

Solution:

$$\begin{aligned} = & \lim_{x \to a} \frac{x - a}{x^2 - a^2} \\ = & \lim_{x \to a} \frac{x - a}{(x - a)(x + a)} \\ = & \lim_{x \to a} \frac{1}{x + a} \\ = & \frac{1}{2a} \end{aligned}$$

Exercises

Find the following limits,

1. $$\lim_{x\to 0} \frac{\sin x}{x}$$

2. $$\lim_{t\to 1} \frac{5t^4 - 4t^2 - 1}{10 - t - 9t^3}$$

3. $$\lim_{x\to \infty} \frac{e^x}{x^2}$$

4. $$\lim_{x\to 0^+} x \ln x$$

5. $$\lim_{x\to -\infty} xe^x$$

6. $$\lim_{x\to \infty} x^{1/x}$$

7. $$\lim_{x\to 2} \frac{x^2 + x + 6}{x^2 + 4}$$

8. $$\lim_{x\to -\infty} \frac{e^{2x}}{x^3}$$

9.
$$\lim_{x\to\infty} \frac{x+\cos x}{x}$$

10.
$$\lim_{x\to\infty} \frac{e^x}{x^k}$$