Definition of a Limit

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Definition of a Limit

In words the *limit* of a function is a value that the function approaches as the independent variable, usally denoted by x, tends to a particular value. This particular value is either a finite value a say, or infinity, ∞. Below is the definition of a *limit* in mathemtical notation,

$$\lim_{x \to a} f(x) = L,$$

where L exists, means,

$$\forall \epsilon > 0, \quad \exists \delta > 0 \ni, \ |x - a| < \delta \implies |f(x) - L| < \epsilon \tag{1}$$

The notation $x \to a$ means that x is approaching a from any direction. However, in some cases it may be the case that the limit only exists when a is being approached from either the left or right. In these cases the limit is defined below,

Right hand Limit Definition

$$\lim_{x \to a^+} f(x) = L$$

is defined as,

$$\forall \epsilon > 0, \quad \exists \delta > 0 \ni, \ x - a > \delta \implies |f(x) - L| < \epsilon \tag{2}$$

In this above limit, x is approaching a from the positive side of a or from the right hand side. Now let's consider the definition of a limit when x is approaching a from the left hand side.

Left hand limit Definition

$$\lim_{x \to a^-} f(x) = L$$

is defined as,

$$\forall \epsilon > 0, \quad \exists \delta > 0 \ni, \ x - a < \delta \implies |f(x) - L| < \epsilon \tag{3}$$

So far we've considered limits as x approaches a finite value a. I some cases the limit occurs when x approaches infinity. Let's take at look at the definition of the limit as x tends to positive infinity and then negative infinity.

Limit as x tends to infinity (Horizontal asymptotes)

$$\lim_{x \to +\infty} f(x) = L$$

is defined by,

$$\forall \epsilon > 0 \ \ \exists N > 0 \ni, \ x > N \Rightarrow |f(x) - L| < \epsilon \tag{4}$$

and,

$$\lim_{x \to -\infty} f(x) = L$$

is defined by,

$$\forall \epsilon > 0 \ \ \exists N < 0 \ni, \ x < N \Rightarrow |f(x) - L| < \epsilon \tag{5}$$

So far we have considered finit limits L. Let's now consider infinte limites, that is when our L is replaced by negative or positive infinity.

Infinite Limits

$$\lim_{x \to a} f(x) = +\infty$$

is defined a

$$\forall N > 0, \ \ \exists \delta > 0 \ni, \ |x - a| < \delta \Rightarrow f(x) > N \tag{6}$$

or

$$\lim_{x \to a} f(x) = -\infty$$

is defined a

$$\forall N < 0, \ \ \exists \delta > 0 \ni, \ |x - a| < \delta \Rightarrow f(x) < N \tag{7}$$

Above we considered the situation where x approaches a from any direction, left or right. However, we can also have x approaching a from just the left or just the right. These definitions are below.

$$\lim_{x \to a^+} f(x) = +\infty$$

is defined as,

$$\forall N > 0, \ \exists \delta > 0 \ni, \ x - a > \delta \Rightarrow f(x) > N \tag{8}$$

and

$$\lim_{x \to a^-} f(x) = +\infty$$

is defined as,

$$\forall N > 0,\ \exists \delta > 0 \ni,\ x - a < \delta \Rightarrow f(x) > N \tag{9}$$

And for the negative limit,

$$\lim_{x \to a^+} f(x) = -\infty$$

is defined as,

$$\forall N < 0,\ \exists \delta > 0 \ni,\ x - a > \delta \Rightarrow f(x) < N \tag{10}$$

and

$$\lim_{x \to a^-} f(x) = -\infty$$

is defined as,

$$\forall N < 0,\ \exists \delta > 0 \ni,\ x - a < \delta \Rightarrow f(x) < N \tag{11}$$

Examples

Find the following limit.

$$\lim_{x \to 0} \frac{x^4 - 5x}{x^3 + 2x}$$

Solution:

$$\begin{aligned} & \lim_{x \to 0} \frac{x^4 - 5x}{x^3 + 2x} \\ = & \lim_{x \to 0} \frac{x(x^3 - 5)}{x(x^2 + 2)} \\ = & \lim_{x \to 0} \frac{x^3 - 5}{x^2 + 2} \\ = & \frac{0 - 5}{x + 2} \\ = & -\frac{5}{2} \end{aligned}$$

Therefore,

$$\lim_{x \to 0} \frac{x^4 - 5x}{x^3 + 2x} = -\frac{5}{2}.$$

Examples

Find the limit below,

$$\lim_{x\to\infty} \frac{3x^2 - 2x + 1}{5x^3 + 4}$$

Solution:

$$\begin{aligned}
& \lim_{x\to\infty} \frac{3x^2 - 2x + 1}{5x^3 + 4} \\
= & \lim_{x\to\infty} \frac{x^3\left(3 - 2\frac{1}{x^2} + \frac{1}{x^3}\right)}{x^3\left(5 + 4\frac{1}{x^3}\right)} \\
= & \lim_{x\to\infty} \frac{3 - \frac{2}{x^2} - \frac{1}{x^2}}{5 + \frac{4}{x^3}} \\
= & \lim_{x\to\infty} \frac{3 - 0 - 0}{5 + 0} \\
= & \frac{3}{5}
\end{aligned}$$

Therefore,

$$\lim_{x\to\infty} \frac{3x^2 - 2x + 1}{5x^3 + 4} = \frac{3}{5}$$

Exercises

Find the following limits,

1. $$\lim_{x\to\infty} \frac{3x^2 - 2x + 1}{5x^2 + 4}$$

2. $$\lim_{x\to\infty} \frac{3 - x^2}{x^3 - x^2 + 1}$$

3. $$\lim_{x\to 3} \frac{x - 3}{x^2 - 9}$$

4. $$\lim_{x\to -1} \frac{x^2 - 2x - 3}{x + 1}$$

5. $$\lim_{x\to 0} \frac{x^3 - 5x}{x^2 + 2x}$$

6. $$\lim_{x\to 2} \frac{x - 2}{x + 2}$$

7. $$\lim_{x\to -3} \frac{x - 1}{x^2 + 2x - 3}$$