Complex Numbers

Raise My Marks

RaiseMyMarks.com

2020

Complex Numbers

The most basic complex number is i. i is defined as,

$$i^2 = -1 \text{ or } i = \sqrt{-1} \tag{1}$$

This additional number now allows us to factor *every* quadratic and polynomial equation. The discriminant is no longer a factor since with this new number $i = \sqrt{-1}$ we can take the square root of a negative number.

A general complex number can be written as,

$$\begin{aligned} z &= a + ib, \text{ where } a, b \in \mathbb{R} && (2) \\ &= \mathbb{R}e(z) + i\mathbb{I}m(z) && (3) \end{aligned}$$

where $\mathbb{R}e(z) = a$ is called the *real* part of z and $\mathbb{I}m(z) = b$ is called the *imaginary* part of z. Now a few definitions related to complex numbers.

Definitions related to complex numbers

The **conjugate** of $z = a + ib$ is given by,

$$\overline{z} = a - ib$$

The **modulus** of $z = a + ib$ is given by,

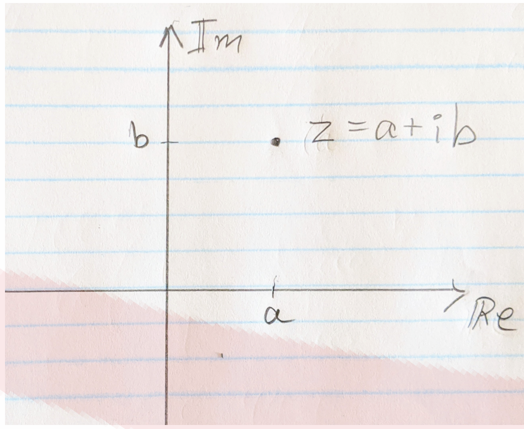
$$\begin{aligned} |z| &= \sqrt{z\overline{z}} \\ &= \sqrt{(a+ib)(a-ib)} \\ &= \sqrt{a^2 - abi + abi - b^2i^2} \\ &= \sqrt{a^2 - b^2(-2)} \\ \therefore |z| &= \sqrt{a^2 + b^2} \end{aligned}$$

Graphing complex numbers

The Cartesian plane is a plane of real numbers. With the addition of complex numbers we now have a complex plane. The complex plane is similar to the Cartesian plane except now the vertical axis represents the imaginary part of a complex number and the horizontal axis represents the real part of the complex number. This means, that each complex number $z = a + ib$ can be represented by an ordered pair,

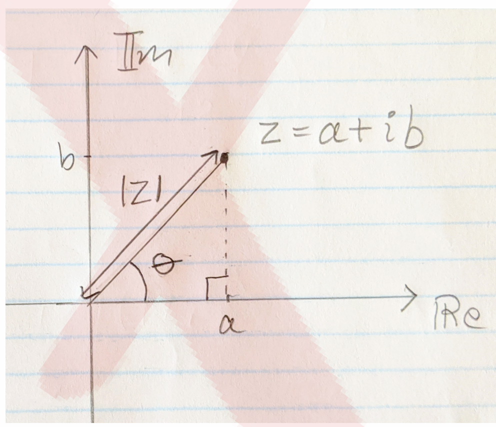
$$(a, b) = (\mathbb{R}e(z), \mathbb{I}m(z)).$$

This point and representation of the complex number z can be plotted on the complex plane just the way you would plot an ordered pair (x, y).



Polar Coordinates

Another representation of a complex number $z = a + ib$ is through somthing called *polar coordinates*. Now that we know how to plot a complex number in the complex plane, let's take a closer look at this point.



Notice that the point $z = a + ib$ lies on a circle of radius r. Notice that the radius equals the modulus of the complex number.

$$r = |z| = \sqrt{a^2 + b^2}$$

If we take the positive real axis as $0°$ and the counter clockwise direction as a positive angle measure, then the angle θ is the angle of rotation along the circumference of the circe of radius $|z|$ where the complex number $z = a + ib$. Using the theorm of Pythagorus and trigonometric ratios we can express a and b in terms of $|z|$ and θ. Now we have,

$$a = |z| \cos\theta, \; b = |z| \sin\theta \tag{4}$$

Now we can rewrite z as,

$$\begin{aligned} z &= a + ib \\ &= |z|\cos\theta + i|z|\sin\theta \\ &= |z|(\cos\theta + i\sin\theta) \end{aligned}$$

If we let,

$$r = |z| \text{ and } e^{i\theta} = \cos\theta + i\sin\theta$$

we have the following representation of our complex number z.

$$z = re^{i\theta}$$

This is *polar coordinate* representatio of the complex number $z = a+ib$ where $r = |z|$ is the modulus of z and θ is called the *argument* of the complex number z.

Properties of complex numbers

Most of the regular properties of arithmetic learned for real numbers hold for complex numbers. Let's let z and w be two complex numbers $z = a + ib$ and $w = c + id$. And let's let α is any real number. Then we have the following,

1. $z + w = w + z$
2. $z + (w + t) = (z + w) + t$ where t is a complex number
3. $zw = wz$
4. $(zw)t = z(wt)$ where t is a complex number
5. $\alpha\mathbb{R}e(z) = \mathbb{R}e(\alpha z)$ and $\alpha\mathbb{I}m(z) = \mathbb{I}m(\alpha z)$
6. $\mathbb{R}e(z) = \mathbb{R}e(\overline{z})$ and $\mathbb{I}m(z) = -\mathbb{I}m(\overline{z})$
7. $\overline{(zw)} = (\overline{z})(\overline{w})$
8. $\overline{\overline{z}} = z$
9. $\overline{\alpha z} = \alpha\overline{z}$

Exercises

1. Given the coplex numbers $z = 3 + 4i,\ w = -7 + 2i,\ u = 5 - 6i$, evaluate the following,

   a) $\overline{w}z$

   b) $\left|\frac{\overline{u}}{|\overline{u}|}\right|$

2. Graph the following complex number on the same complex plane.

   a) $6 + 2i$

   b) $\frac{4-3i}{\overline{4-3i}}$

3. Rewrite the following in polar form.

   a) $3\sqrt{3} - 3i$

   b) $-4 - 4i$

4. Find the modulus for the following complex numbers,

   a) $12i + 2$

   b) $100e^{\pi i/6}$

5. What is the real and imaginary part of the following complex numbers?

   a) $\frac{2}{5}e^{7\pi i/12}$

   b) $4e^{4\pi i/3}$