

Complex Numbers

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## Complex Numbers

The most basic complex number is  $i$ .  $i$  is defined as,

$$i^2 = -1 \text{ or } i = \sqrt{-1} \quad (1)$$

This additional number now allows us to factor *every* quadratic and polynomial equation. The discriminant is no longer a factor since with this new number  $i = \sqrt{-1}$  we can take the square root of a negative number.

A general complex number can be written as,

$$z = a + ib, \text{ where } a, b \in \mathbb{R} \quad (2)$$

$$= \operatorname{Re}(z) + i\operatorname{Im}(z) \quad (3)$$

where  $\operatorname{Re}(z) = a$  is called the *real* part of  $z$  and  $\operatorname{Im}(z) = b$  is called the *imaginary* part of  $z$ . Now a few definitions related to complex numbers.

### Definitions related to complex numbers

The **conjugate** of  $z = a + ib$  is given by,

$$\bar{z} = a - ib$$

The **modulus** of  $z = a + ib$  is given by,

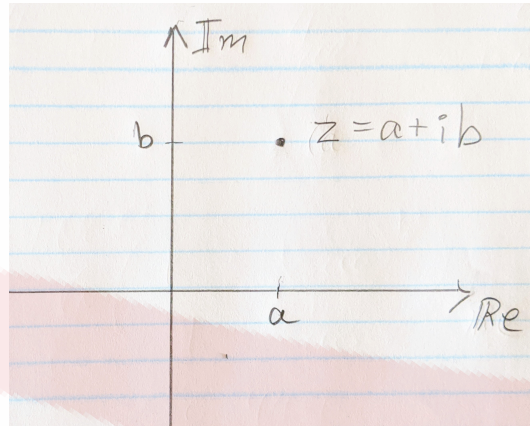
$$\begin{aligned} |z| &= \sqrt{z\bar{z}} \\ &= \sqrt{(a + ib)(a - ib)} \\ &= \sqrt{a^2 - abi + abi - b^2i^2} \\ &= \sqrt{a^2 - b^2(-2)} \\ \therefore |z| &= \sqrt{a^2 + b^2} \end{aligned}$$

### Graphing complex numbers

The Cartesian plane is a plane of real numbers. With the addition of complex numbers we now have a complex plane. The complex plane is similar to the Cartesian plane except now the vertical axis represents the imaginary part of a complex number and the horizontal axis represents the real part of the complex number. This means, that each complex number  $z = a + ib$  can be represented by an ordered pair,

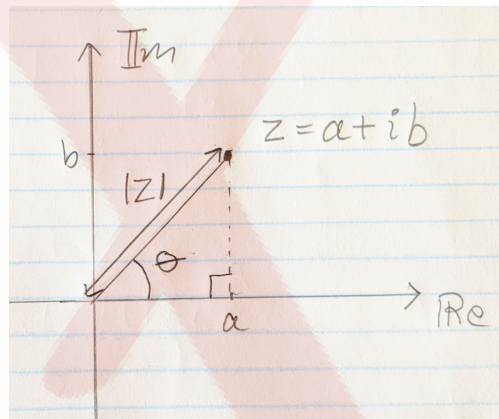
$$(a, b) = (\operatorname{Re}(z), \operatorname{Im}(z)).$$

This point and representation of the complex number  $z$  can be plotted on the complex plane just the way you would plot an ordered pair  $(x, y)$ .



### Polar Coordinates

Another representation of a complex number  $z = a + ib$  is through something called *polar coordinates*. Now that we know how to plot a complex number in the complex plane, let's take a closer look at this point.



Notice that the point  $z = a + ib$  lies on a circle of radius  $r$ . Notice that the radius equals the modulus of the complex number.

$$r = |z| = \sqrt{a^2 + b^2}$$

If we take the positive real axis as  $0^\circ$  and the counter clockwise direction as a positive angle measure, then the angle  $\theta$  is the angle of rotation along the circumference of the circle of radius  $|z|$  where the complex number  $z = a + ib$ . Using the theorem of Pythagoras and trigonometric ratios we can express  $a$  and  $b$  in terms of  $|z|$  and  $\theta$ . Now we have,

$$a = |z| \cos \theta, \quad b = |z| \sin \theta \quad (4)$$

Now we can rewrite  $z$  as,

$$\begin{aligned} z &= a + ib \\ &= |z| \cos \theta + i|z| \sin \theta \\ &= |z|(\cos \theta + i \sin \theta) \end{aligned}$$

If we let,

$$r = |z| \text{ and } e^{i\theta} = \cos \theta + i \sin \theta$$

we have the following representation of our complex number  $z$ .

$$z = re^{i\theta}$$

This is *polar coordinate* representatio of the complex number  $z = a + ib$  where  $r = |z|$  is the modulus of  $z$  and  $\theta$  is called the *argument* of the complex number  $z$ .

### Properties of complex numbers

Most of the regular properties of arithmetic learned for real numbers hold for complex numbers. Let's let  $z$  and  $w$  be two complex numbers  $z = a + ib$  and  $w = c + id$ . And let's let  $\alpha$  is any real number. Then we have the following,

1.  $z + w = w + z$
2.  $z + (w + t) = (z + w) + t$  where  $t$  is a complex number
3.  $zw = wz$
4.  $(zw)t = z(wt)$  where  $t$  is a complex number
5.  $\alpha \operatorname{Re}(z) = \operatorname{Re}(\alpha z)$  and  $\alpha \operatorname{Im}(z) = \operatorname{Im}(\alpha z)$
6.  $\operatorname{Re}(z) = \operatorname{Re}(\bar{z})$  and  $\operatorname{Im}(z) = -\operatorname{Im}(\bar{z})$
7.  $\overline{(zw)} = (\bar{z})(\bar{w})$
8.  $\overline{\bar{z}} = z$
9.  $\overline{\alpha z} = \alpha \bar{z}$

## Exercises

1. Given the complex numbers  $z = 3 + 4i$ ,  $w = -7 + 2i$ ,  $u = 5 - 6i$ , evaluate the following,

a)  $|w|$

b)  $3z - 2u$

2. Graph the following complex number on the same complex plane.

a)  $1 + i$

b)  $-2 - i$

3. Rewrite the following in polar form.

a)  $\sqrt{3} + i$

b)  $-1 + i$

4. Find the modulus for the following complex numbers,

a)  $\frac{z}{|z|}$

b)  $\sqrt{3} + i$

5. What is the real and imaginary part of the following complex numbers?

a)  $2e^{2\pi/3}$

b)  $3e^{7\pi i/4}$