Symmetric equation of a line in $\mathbb{R}^3$

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Equation of a line in $\mathbb{R}^3$

Let's consider 3-dimensional space, $\mathbb{R}^3$, and the forms of the equation of a line.

Vector equation of a line

First let's recall the **vector form of the equation of a line** in $\mathbb{R}^3$. This is similar to the vector form of the equation of a line in the Cartesian plane. We have a point $\underline{x}_0 = (x_{01}, x_{02}, x_{03})$ the lines goes through, and a vector $\underline{m} = (m_1, m_2, m_3)$ for the direction of the line.

$$\underline{x} = \underline{x}_0 + t\underline{m}, t \in \mathbb{R}$$

is the fector form of the equation of a line in $\mathbb{R}^3$.

Parametric equation of a line

The **parametric equation of a line** in $\mathbb{R}^3$ is again similar to the parametric equation of a line in $\mathbb{R}^2$. Start with the vecor form and rewrite it componentwise to give the parametric eqation of the line.

$$x_1 = x_{01} + tm_1 \quad (1)$$
$$x_2 = x_{02} + tm_2 \quad (2)$$
$$x_3 = x_{03} + tm_3 \quad (3)$$

where $t \in \mathbb{R}$.

Symmtric equation of a line in $\mathbb{R}^3$

The **symmetric equation of a line** in $\mathbb{R}^3$ is not considered in the Cartesian plane but in 3-dimensional space $\mathbb{R}^3$. Given a point $\underline{x}_0 = (x_0, y_0, z_0)$ on the line and the direction of the line $\underline{m} = (a, b, c) \neq (0, 0, 0)$ the symmetric equation of the line is given by,

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

We can get these equations, by taking the 3 equations of the parametric equation of a line and solve each for t. So, from equations 1, 2 and 3 above, we have,

$$x_1 = x_{01} + tm_1 \implies \frac{x_1 - x_{01}}{m_1} = t \quad (4)$$
$$x_2 = x_{02} + tm_2 \implies \frac{y_1 - y_{01}}{m_2} = t \quad (5)$$
$$x_3 = x_{03} + tm_3 \implies \frac{z_1 - z_{01}}{m_3} = t \quad (6)$$

Equating these three equations, 4, 5, and 6 we have,

$$\frac{x_1 - x_{01}}{m_1} = \frac{y_1 - y_{01}}{m_2} = \frac{z_1 - z_{01}}{m_3}$$

which is exactly the symmetric equation of the line through the point $\underline{x}_0 = (x_{01}, y_{01}, z_{01})$ in the direction $\underline{m} = (m_1 m_2, m_3)$ where $\underline{x} = (x_1, y_1, z_1)$ is a general point on this line.

Examples

Find the symmetric equation of the line through P and in the direction QR where P=(3,1,2), Q=(-1,2, 1) and R=(2, 3, -1).

Solution: Let's find the direction $\underline{m}$.

$$\underline{m} = R - Q = (2,3,-1) - (-1,2,1) = (3,1,-2)$$

Let's find the parametric equation of the line through P now.

$$\begin{aligned} \underline{x} &= P + t\underline{m} \\ &= (3,1,2) + t(3,1,-2) \\ \Longrightarrow \quad & x = 3 + 3t, y = 1 + t, z = 2 - 2t, \end{aligned}$$

where t is any realy number or $t \in \mathbb{R}$. Now, let's solve for t.

$$\frac{x-3}{3} = t, y - 1 = t, \frac{z-2}{-2} = t$$

Therefore, our symmetric equation for the line is,

$$\frac{x-3}{3} = y - 1 = \frac{x-2}{2}.$$

Exercises

Given the points P, Q and R below determine the symmetric equation of the line through the point P in the direction of QR.

c) $P(-1, 2, 1), Q(2, 0, -1), R(-1, 3, 0)$

h) $P(-1, -2, 1), Q(4, 3, -1), R(3, 1, 2)$

d) $P(3, -2, 0), Q(2, 2, 2), R(-1, 2, -1)$

j) $P(0, -4, -1), Q(3, 0, 0), R(-2, 3, 1)$

g) $P(4, 0, -3), Q(-1, 2, 0), R(1, 1, 1)$

k) $P(-2, 5, -3), Q(2, 2, -2), R(1, 3, 4)$