Completing the Square

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2020

What does "completing the square" mean?

First we need to understand what a "square" is. A square is a quadratic that is a *perfect square*. So it can be factored in the following way,

$$(x+a)^2 \tag{1}$$

where a is any real number. If we expand or multiply out (1) we can see what a complete square looks like.

$$\begin{aligned}(x+a)^2 &= (x+a)(x+a)\\ &= x^2+ax+ax+a^2\\ &= x^2+2ax+a^2 \text{ which is a perfect square.}\end{aligned}$$

To "complete the square" means we're given some part of

$$x^2+2ax+a^2 \tag{2}$$

and we have to "complete" it so we have

$$x^2+2ax+a^2 = (x+a)^2$$

somewhere in our quadratic. Why we want to complete the square will come later when we look at graphing of quadratics.

Example

Let's consider an example and go through the process of completing the square. Complete the square of the following quadratic,

$$x^2+6x$$

Solution What do we need to add to this to complete the square?

1. First, we know that $6=2a$. This means that $a=3$.
2. Second, we have to add $a^2=3^2=9$.

3. Third, we now have,

$$x^2 + 6x + 9 - 9$$

Notice that we added 9 and subtracted 9. The reason for this is we cannot just add a number to a function. If we did then we would be changing the function. So instead we add a 0. How we write 0 will make all the difference when completing the square. In this case since we need to add 9, we write 0 as,

$$a^2 - a^2 \text{ or } 9 - 9$$

4. Fourth, rewrite with a perfect square in the expression of the quaddratic. Once we add our strategicly written 0 we have,

$$\begin{aligned} x^2 + 6x + 9 - 9 &= (x^2 + 6x + 9) - 9 \\ &= (x + 3)(x - 3) - 9 \\ &= (x + 3)^2 - 9 \end{aligned}$$

Example

Let's try another example. Complete the square of the following quadratic,

$$4x^2 + 7x.$$

Solution

1. **What part of (2) is missing or needs to be altered?** We need to work with the form $x^2 + bx$. So let's factor out the 4 to get us into the required form.

$$4x^2 + 7x = 4\left(x^2 + \frac{7}{4}x\right)$$

2. **Now, complete the square of** $x^2 + \frac{7}{4}x$ This means,

$$\begin{aligned} \frac{7}{4} &= 2a \\ \frac{7}{8} &= a \end{aligned}$$

3. **Add** 0. We will be adding zero where our 0 is written as,

$$0 = a^2 - a^2 = \left(\frac{7}{8}\right)^2 - \left(\frac{7}{8}\right)^2 = \frac{49}{64} - \frac{49}{64}$$

This then gives us,

$$4\left(x^2+\frac{7}{4}x+\frac{49}{64}-\frac{49}{64}\right)$$

4. **Rewrite the original equation with as a completed square.** Let's give this a try.

$$\begin{aligned} 4\left(x^2+\frac{7}{4}x+\frac{49}{64}\right)-4\left(\frac{49}{64}\right) &= 4\left(x+\frac{7}{8}\right)\left(x+\frac{7}{8}\right)-\frac{49}{16} \\ &= 4\left(x+\frac{7}{8}\right)^2=\left(\frac{7}{4}\right)^2 \end{aligned}$$

Example

Let's try another example. Complete the square of the following function,

$$x^2+8x-14$$

1. Is this a completed square? No
2. Identify the part that will be completed as a square. In this case it is x^2+8x will be completed as a square.
3.
$$\begin{aligned} 8 &= 2a \\ 4 &= a \end{aligned}$$
4. Add 0 where
$$0=a^2-a^2=16-16$$
This gives us,
$$(x^2+8x)-14 = (x^2+9x+16-16)-14$$
5. Rewrite function as a completed square.
$$\begin{aligned} (x^2+8x)-14 &= (x^2+9x+16-16)-14 \\ &= (x^2+8x+16)-16-14 \\ &= (x+4)(x+4)-30 \\ &= (x+4)^2-30 \end{aligned}$$

Exercises

Complete the square of the following quadratics.

a) $x^2 - 12x$

b) $2x^2 - 7x + 10$

c) $3x^2 - 4x$

d) $-6x^2 + 20x$

e) $x^2 + 5x - 10$

f) $x^2 + 8x$

g) $-4x^2 + 8x - 12$